

AM214-2023: LECTURE 34

LECTURE 34

THE SVD - CONTINUED

1

$A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = r$

$$A = U \Sigma V^T$$

$\begin{matrix} | & & | \\ \hline u_1 & u_2 & u_3 \\ \hline | & & | \end{matrix}$

$m \times n$

$\begin{matrix} | & & | \\ \hline \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \sigma_2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \sigma_3 & 0 & 0 & 0 \\ \hline & & & 0 & & \end{matrix}$

$m \times n$

$\begin{matrix} \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \end{matrix}$

V^T

- ⊙ Rank-one decomp.
- ⊙ Reduced SVD
- ⊙ Application: Images
- ⊙ Fundamental subspaces
- ⊙ Moore-Penrose inverse

2

$$= \begin{bmatrix} | & | & | \\ \hline u_1 & u_2 & u_3 \\ \hline | & & | \end{bmatrix} \begin{bmatrix} \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \end{bmatrix}$$

$$= u_1 \sigma_1 v_1^T + u_2 \sigma_2 v_2^T + u_3 \sigma_3 v_3^T$$

$$= \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T$$

$$= \begin{bmatrix} | & | \\ \hline u_1 & u_2 \\ \hline | & | \end{bmatrix} \begin{bmatrix} \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \end{bmatrix}$$

$$= \begin{bmatrix} | & | \\ \hline u_1 & u_2 \\ \hline | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \end{bmatrix}$$

\bar{U}

Σ

\bar{V}^T

← reduced SVD

3

$$(3 \times 6) \quad (2 \times 6) = (3 \times 6)$$

$A \in \mathbb{R}^{m \times n}$

$$= \bar{U} \bar{\Sigma} \bar{V}^T$$

$\uparrow \quad \uparrow \quad \uparrow$
 $(m \times r) \quad (r \times r) \quad (r \times n)$

\bar{U}, \bar{V} are reduced orthogonal matrices.

lossless / lossy compression

$$A = \begin{bmatrix} 251 & 256 & & & & \\ & & & & & \\ & & & 10 & 12 & \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix}$$

4

$$A = 45000 u_1 v_1^T + 6300 u_2 v_2^T$$

How much saved (40 of the 257 singular values)

260×40	257×40	40
~~~~~	~~~~~	
col's of U	col's of V	

# THE FUNDAMENTAL SUBSPACES

$$A \in \mathbb{R}^{m \times n} \quad \boxed{5}$$

SPACE	LIES IN	DIM	ORTHOGON
Row	$\mathbb{R}^n$	$r$	↙
Null	$\mathbb{R}^n$	$n-r$	
Col	$\mathbb{R}^m$	$r$	↘
Left null	$\mathbb{R}^m$	$m-r$	



$$A = U \Sigma V^T \quad \boxed{6}$$

$$\begin{bmatrix} \vdots \\ \underline{a}_1 \\ \vdots \\ \underline{a}_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \underline{u}_1 & \underline{u}_2 & \dots \\ \vdots & \vdots & \dots \end{bmatrix} \begin{bmatrix} \sigma_1 \underline{v}_1 & \sigma_1 \underline{v}_2 & \dots \\ \sigma_2 \underline{v}_1 & \sigma_2 \underline{v}_2 & \dots \\ \vdots & \vdots & \dots \end{bmatrix}$$

Col space

$$\underline{a}_1 = \underline{u}_1 \sigma_1 \underline{v}_1 + \underline{u}_2 \sigma_2 \underline{v}_2 + \underline{u}_3 \sigma_3 \underline{v}_3 \dots \underline{u}_r \sigma_r \underline{v}_r$$

$$\underline{a}_2 = \underline{u}_1 \sigma_1 \underline{v}_1 + \dots$$

Each column of  $A$  is a linear combination of  $\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_r\}$

First  $r$  columns of  $U$  is a basis for  $\mathcal{C}(A)$ . 7

Left null sp.

If  $y^T A = \underline{0}^T$ , then  $y \in \mathcal{L}(A)$

$$A = U \Sigma V^T$$

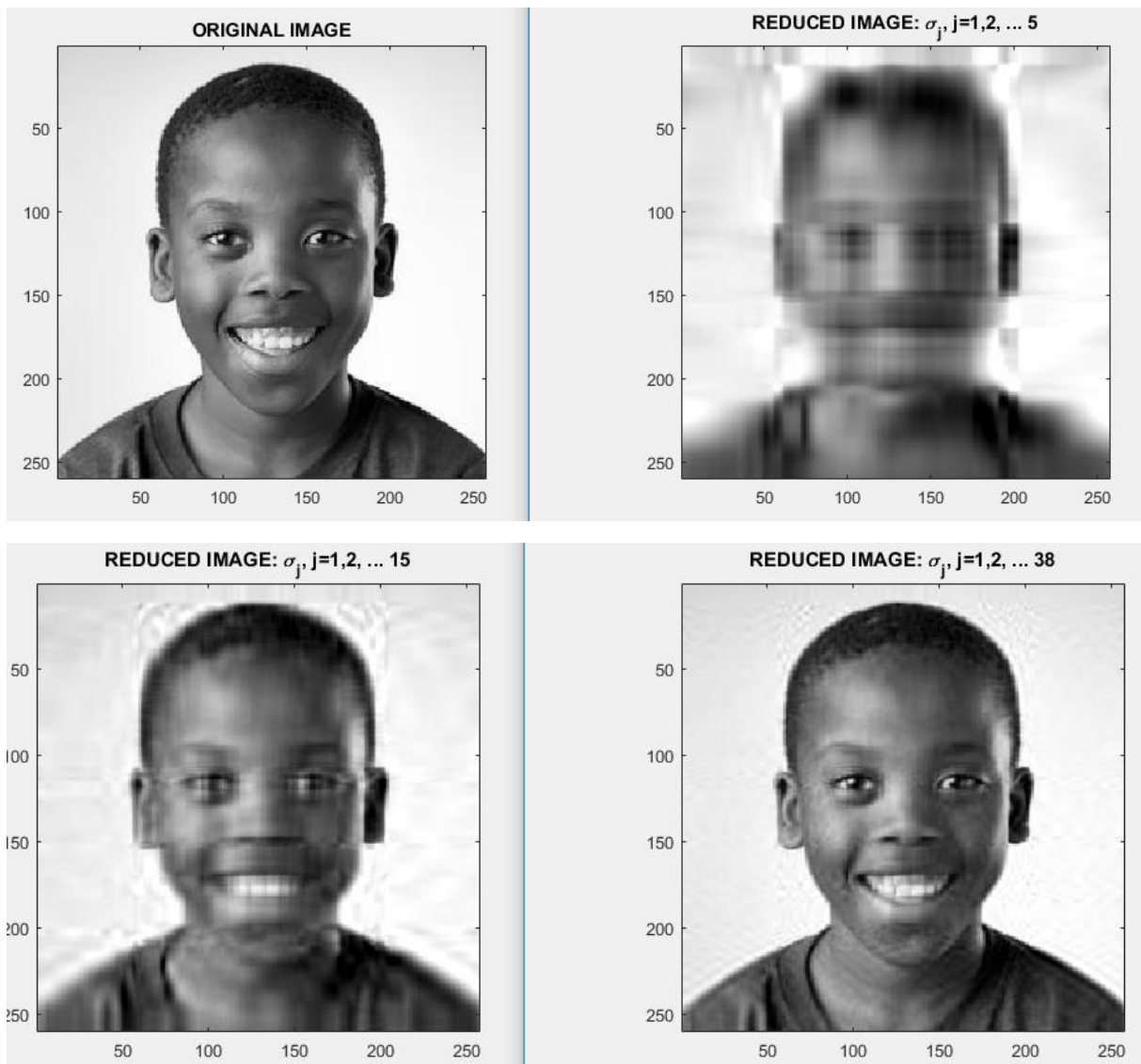
$$U^T A = \Sigma V^T$$

$$\begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_r^T \\ \vdots \\ u_{m-r}^T \\ u_m^T \end{bmatrix} A = \begin{bmatrix} \sigma_1 v_1^T \\ \sigma_2 v_2^T \\ \vdots \\ \sigma_r v_r^T \\ \vdots \\ 0^T \\ 0^T \end{bmatrix}$$

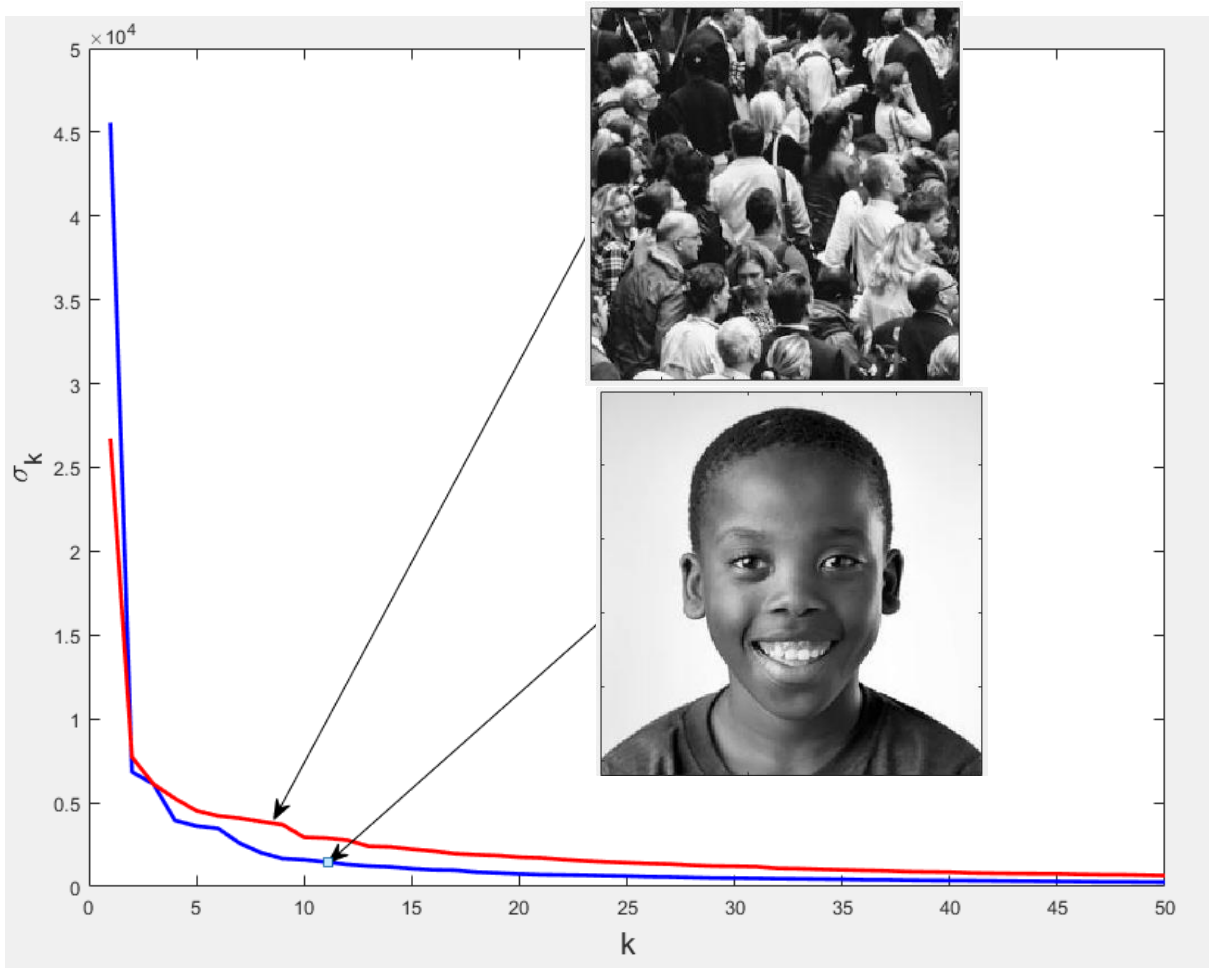
Last  $m-r$  columns of  $U$  is a basis for  $\mathcal{L}(A)$ .

$$\begin{bmatrix} U \\ \mathcal{C}(A) \\ \mathcal{L}(A) \end{bmatrix} \begin{matrix} m \\ m-r \end{matrix} \quad \begin{matrix} r \\ m-r \end{matrix} \quad \begin{bmatrix} V^T \\ R(A) \\ N(A) \end{bmatrix} \quad \begin{matrix} r \\ m-r \end{matrix}$$

8



The image size is  $260 \times 257$ . Total number of pixels is 66820. The last reduction, with 38 singular values kept, needs storage for  $260 \times 38 + 257 \times 38 + 38$  values. That is 19684 values. The data reduction is  $19684/66820 = 29.46\%$ .



The blue graph plots the first 50 singular values of the bottom picture (one person's face) .

The red graph plots the first 50 singular values of the top picture (a crowd) .

The larger singular values for the higher values of k signify a more detailed image.