AM214-2023: LECTURE 33

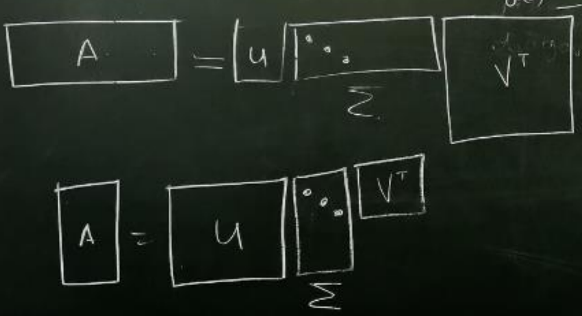
LECTURE 33 THE SVD

1

A is $m\times n$, $A = U\Sigma V^T$

A: $m\times n$; U: $m\times m$; Σ: $m\times n$; V: $n\times n$.

U, V: orthogonal; Σ: diag



- Theory (again)
- Example
- Geometric interp.
- Rank-one decomp.
- Reduced SVD
- Fundamental subspaces
- Moore-Penrose inverse
- Appl: Image compression

2

$A = U\Sigma V^T$

$AA^T = U\Sigma \cancel{V^T V}\Sigma^T U^T$

$= U\,\Sigma\Sigma^T\,U^T$

square sym $= U\Lambda U^T$

$Q \;\; \Lambda \;\; Q^T$

$$\Sigma\Sigma^T = \begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\end{bmatrix}\begin{bmatrix}\sigma_1 & 0\\ 0 & \sigma_2\\ 0 & 0\end{bmatrix}$$

$$= \begin{bmatrix}\sigma_1^2 & 0\\ 0 & \sigma_2^2\end{bmatrix} \overset{\text{call it}}{=:} \Lambda$$

$$\Sigma^T\Sigma = \begin{bmatrix}\sigma_1^2 & 0 & 0\\ 0 & \sigma_2^2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$U := Q, \quad \Sigma = \begin{bmatrix}\sqrt{\lambda_1} & 0 & 0\\ 0 & \sqrt{\lambda_2} & 0\end{bmatrix}$$

$$\begin{bmatrix}\times & \times & \times\\ \times & \times & \times\end{bmatrix}$$

Get V (long way):

$A^TA = V\Sigma^T \cancel{U^T U}\Sigma V^T$

$= V\,\underbrace{\Sigma^T\Sigma}_{\Lambda_2}\,V^T$ — much work, don't do it.

$\boxed{3}$

Get V, (better way)

$$A = U \Sigma V^T$$

$$U^T A = \Sigma V^T$$

$$\begin{bmatrix} \text{———} \\ \text{———} \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} \begin{bmatrix} — \underline{v}_1^T — \\ — \underline{v}_2^T — \\ — \underline{v}_3^T — \end{bmatrix}$$

$$= \begin{bmatrix} — \sigma_1 \underline{v}_1^T — \\ — \sigma_2 \underline{v}_2^T — \end{bmatrix}$$

$$\frac{1}{\sigma_1} \{U^T A\}_{1\text{st row}} = \cancel{\sigma_1}\, \underline{v}_1^T$$

$$\frac{1}{\sigma_2} \{U^T A\}_{2\text{nd row}} = \underline{v}_2^T$$

$$\underline{v}_3 = \underline{v}_1 \times \underline{v}_2$$

$\boxed{4}$

Example: $A = \begin{bmatrix} 9 & 12 & 4 \\ -12 & -16 & 3 \end{bmatrix}$

$$A A^T = \begin{bmatrix} 9 & 12 & 4 \\ -12 & -16 & 3 \end{bmatrix} \begin{bmatrix} 9 & -12 \\ 12 & -16 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 241 & -288 \\ -288 & 409 \end{bmatrix}$$

$$\lambda^2 - 650\lambda + 15625 = 0$$

$$\frac{650 \pm 600}{2}$$

$\lambda_1 = 625$

$$\begin{bmatrix} -384 & -288 \\ \cancel{-288} \overset{0}{} & \cancel{-216} \overset{0}{} \end{bmatrix} \begin{bmatrix} x_1 \\ \mu \end{bmatrix}$$

$$\lambda_1 = 625, \quad \lambda_2 = 25$$

$$\sigma_1 = 25, \quad \sigma_2 = 5$$

$$x_1 = -0.75\mu = -\tfrac{3}{4}\mu$$

$$\underline{x}_1 = \begin{bmatrix} +3 \\ -4 \end{bmatrix} \qquad \underline{x}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad U = Q = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

$\boxed{5}$

$$\Sigma = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

$$U^T A = \Sigma V^T$$

$$\frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 9 & 12 & 4 \\ -12 & -16 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 75 & 100 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 20 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} - & \frac{1}{\sigma_1}\underline{v}_1^T & - \\ - & \frac{1}{\sigma_2}\underline{v}_2^T & - \end{bmatrix}$$

$$\underline{v}_1 = \frac{1}{25}\begin{bmatrix} 15 \\ 20 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix} \qquad \underline{v}_2 = \frac{1}{5}\begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\boxed{6}$

$$\underline{v}_1 \times \underline{v}_2 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4-0 \\ 0-3 \\ 0 \end{bmatrix}\frac{1}{5} = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}\frac{1}{5}$$

$$V = \frac{1}{5}\begin{bmatrix} 3 & 0 & 4 \\ 4 & 0 & -3 \\ 0 & 5 & 0 \end{bmatrix}$$

SVD – GEOMETRIC INTERP

2×2

$$A = U \Sigma V^T \qquad \underline{y} = A\underline{x} \text{ for } \|\underline{x}\| = 1$$

$\underline{v}_1$

$$\underline{y} = A\underline{x} = U(\Sigma(V^T \underline{x})) \qquad V^T V$$

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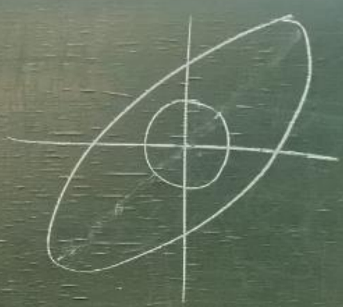
Rank-one decomposition



$$A = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | \\ \underline{u}_1 & \underline{u}_2 & \underline{u}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \underset{\uparrow \atop \sigma_3}{0} & 0 \end{bmatrix} \begin{bmatrix} — & \underline{v}_1^T & — \\ — & \underline{v}_2^T & — \\ — & \underline{v}_3^T & — \\ — & \underline{v}_4^T & — \end{bmatrix}$$

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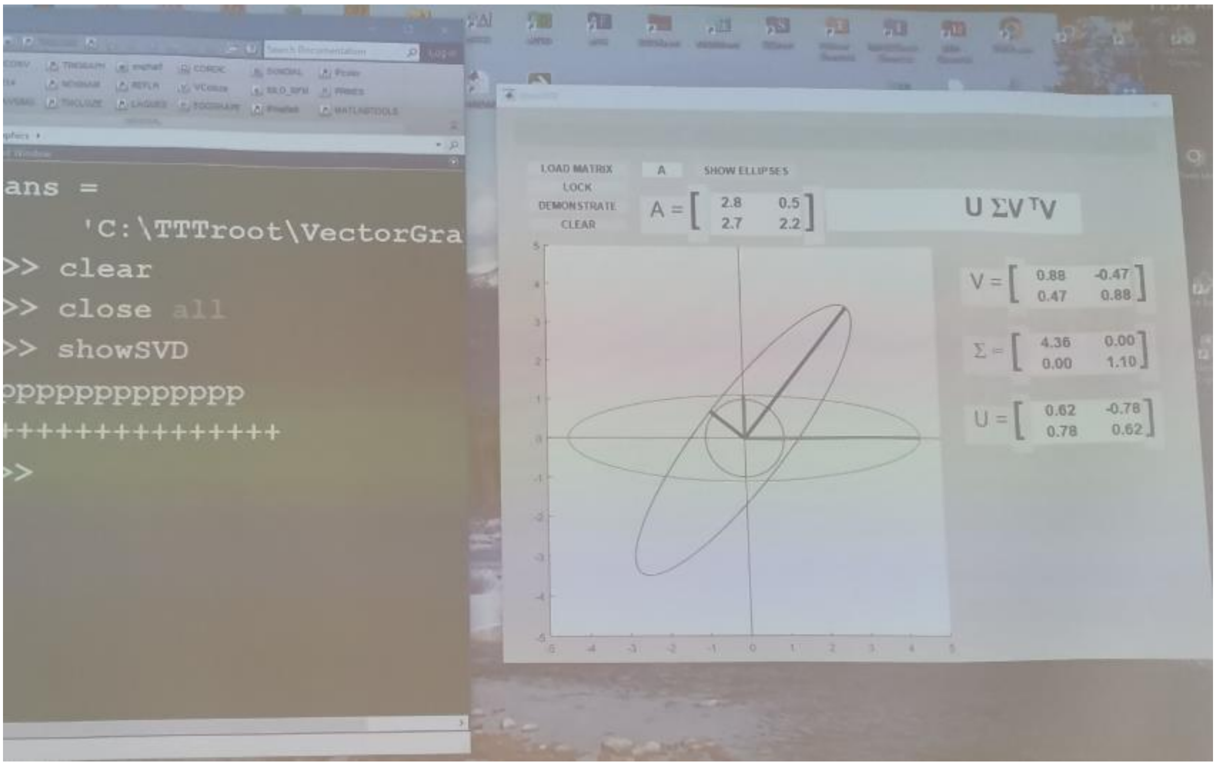
$$= \begin{bmatrix} | & | & | \\ \underline{u}_1 & \underline{u}_2 & \underline{u}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} — & \sigma_1 \underline{v}_1^T & — \\ — & \sigma_2 \underline{v}_2^T & — \\ & 0\ 0\ 0\ 0 & — \end{bmatrix}$$

$$= \underline{u}_1 \sigma_1 \underline{v}_1^T + \underline{u}_2 \sigma_2 \underline{v}_2^T + 0$$

$$= \sigma_1 \underline{u}_1 \underline{v}_1^T + \sigma_2 \underline{u}_2 \underline{v}_2^T$$


ans =
'C:\TTTroot\VectorGra
>> clear
>> close all
>> showSVD
pppppppppppp
++++++++++++++
>>
LOAD MATRIX
LOCK
DEMONSTRATE
CLEAR
A
SHOW ELLIPSES
A = [2.8 0.5; 2.7 2.2]
U ΣV TV
V = [0.88 -0.47; 0.47 0.88]
Σ = [4.36 0.00; 0.00 1.10]
U = [0.62 -0.78; 0.78 0.62]