AM214-2023: LECTURE 32

$\boxed{\text{LECTURE 32}}$ QUADRATIC CURVES
THE SVD

$\boxed{1}$

Example: $4x^2+10xy+4y^2=9$

Find a, b, θ and draw.

$\underline{y}^T A \underline{y} = 9 \qquad \underline{y} = \begin{bmatrix} x \\ y \end{bmatrix}$

$A = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix} \qquad A = Q \Lambda Q^T$

$Q = \begin{bmatrix} + & - \\ + & + \end{bmatrix}$ or $\begin{bmatrix} + & + \\ - & + \end{bmatrix}$

$\begin{vmatrix} 4-\lambda & 5 \\ 5 & 4-\lambda \end{vmatrix}$

$\boxed{2}$

$\lambda^2 - 8\lambda - 9 = 0 \qquad (\lambda+1)(\lambda-9) = 0$

$\lambda_1 = 9, \quad \lambda_2 = -1$

$\underline{\lambda_1 = 9}$:

$\begin{bmatrix} -5 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \mu \end{bmatrix} = 0 \qquad x_1 = \mu \qquad \underline{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

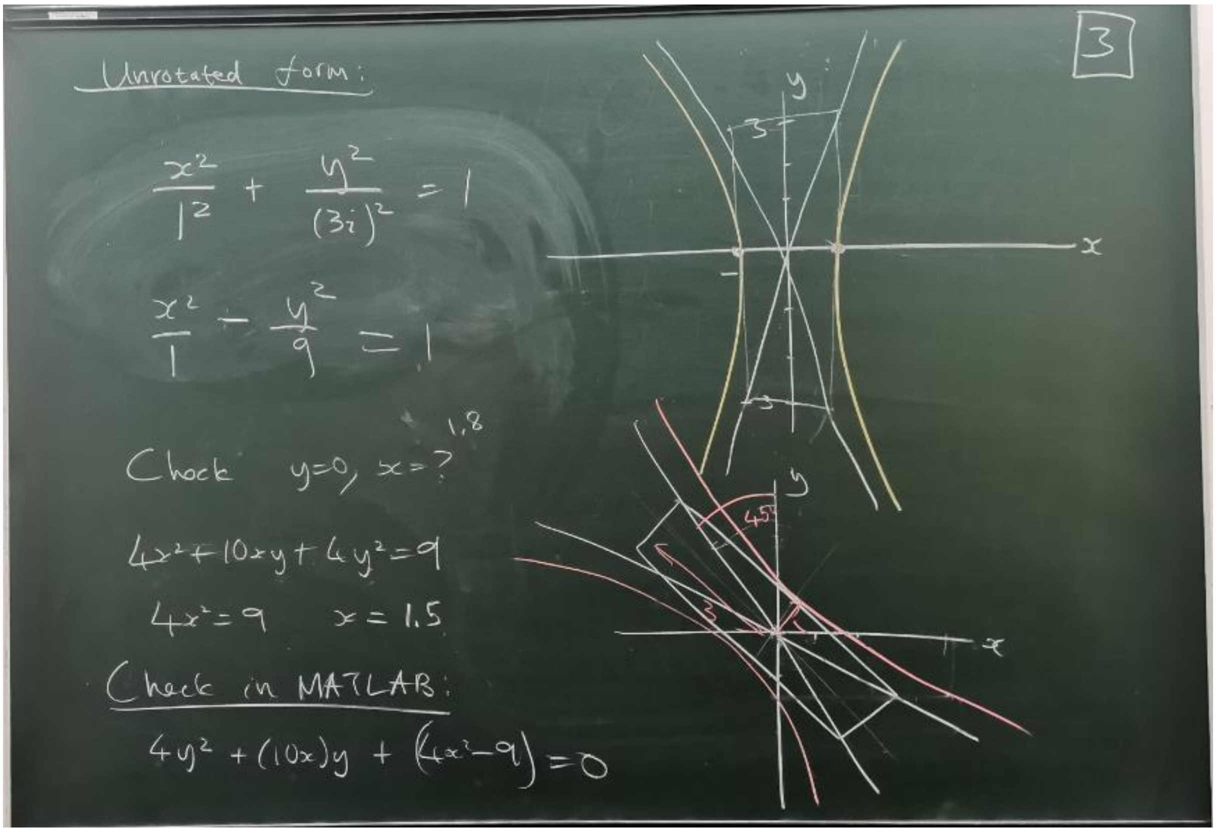
$\underline{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

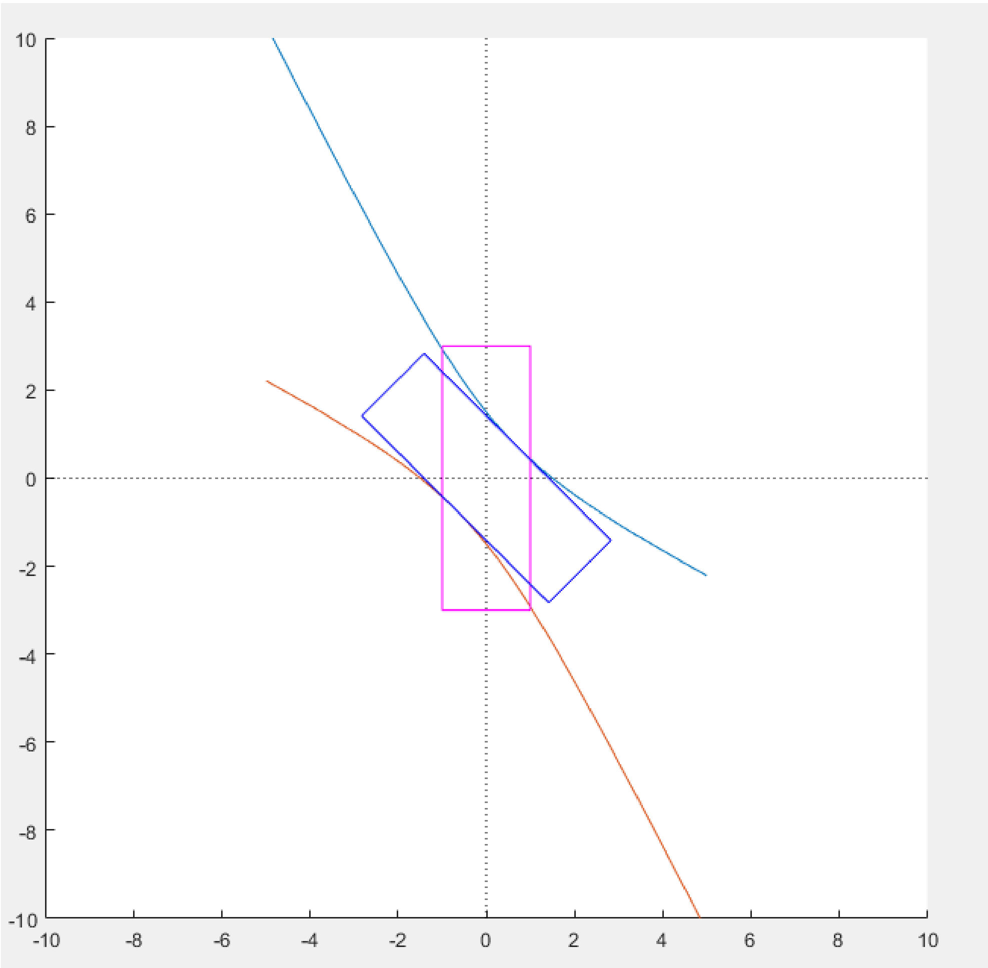
$Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 9 & 0 \\ 0 & -1 \end{bmatrix}$

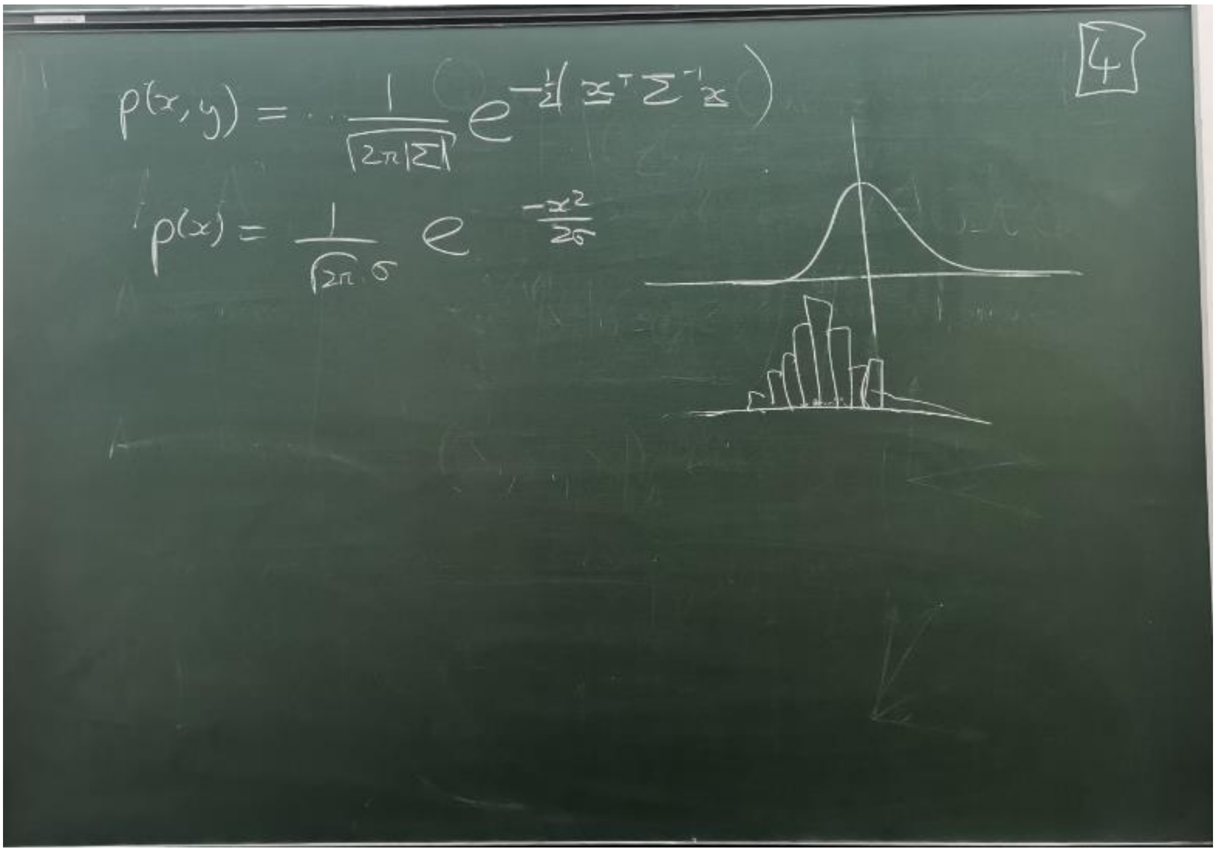
$a = \sqrt{\dfrac{d}{\lambda_1}} \qquad b = \sqrt{\dfrac{d}{\lambda_2}} \qquad -\sin\theta =$ upper-right element of Q.

$a = \sqrt{\dfrac{9}{9}} = 1 \qquad b = \sqrt{\dfrac{9}{-1}} = 3i \qquad -\sin\theta = -\dfrac{1}{\sqrt{2}}$

$\theta = 45^\circ$


3
Unrotated form:
$\frac{x^2}{1^2} + \frac{y^2}{(3i)^2} = 1$
$\frac{x^2}{1} - \frac{y^2}{9} = 1$
Check $y=0$, $x=?$ 1.8
$4x^2+10xy+4y^2=9$
$4x^2=9$ $x=1.5$
Check in MATLAB:
$4y^2+(10x)y+(4x^2-9)=0$
y
x
3
-3
45°


10
8
6
4
2
0
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
0
2
4
6
8
10



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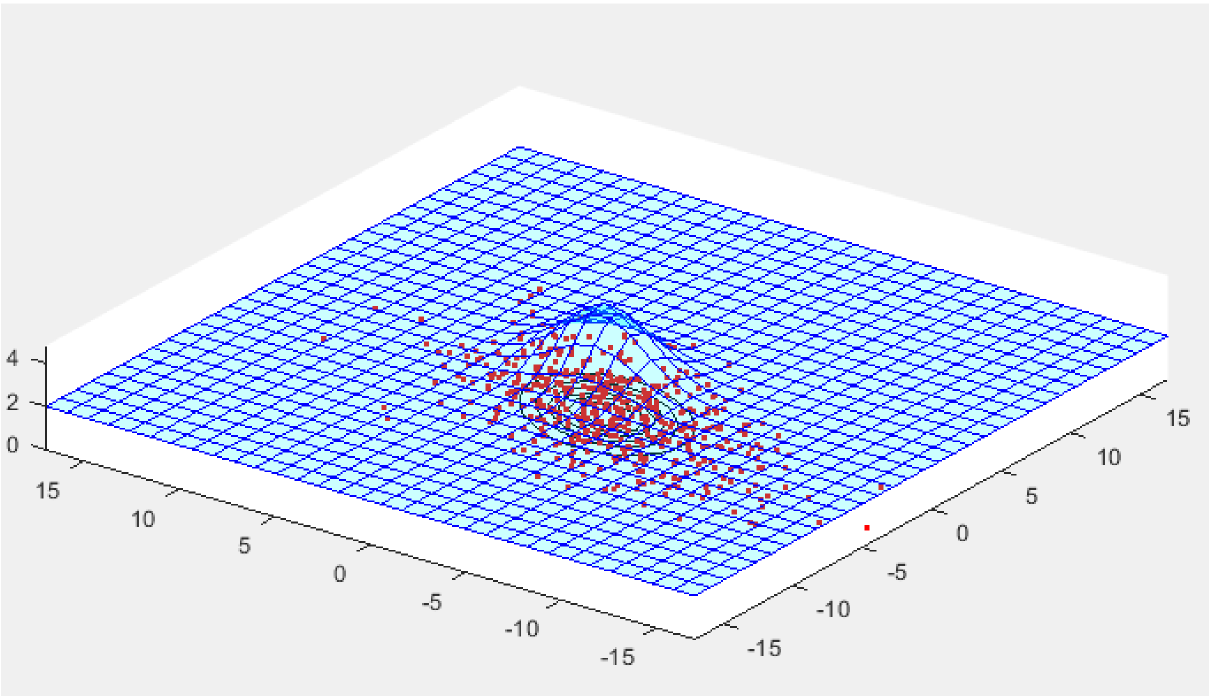


4
2
0
15
10
5
0
-5
-10
-15
15
10
5
0
-5
-10
-15

5

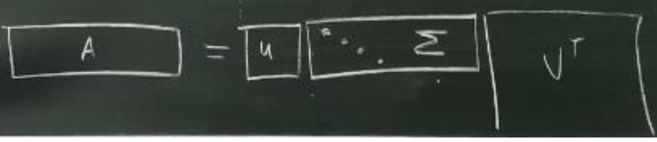
THE SINGULAR VALUE DECOMP.

The SVD for $A \in \mathbb{R}^{m \times n}$

$$\underset{m \times n}{A} = \underset{\substack{\text{orth} \\ m \times m}}{U} \; \underset{m \times n}{\Sigma} \; \underset{\substack{\text{orth.} \\ n \times n}}{V^T}$$

$$2 \times 4 = \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \end{bmatrix}$$

$$3 \times 2 = \begin{bmatrix} x & 0 \\ 0 & x \\ 0 & 0 \end{bmatrix}$$



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2x2 DEMO

$$\begin{bmatrix} x & x \\ x & x \end{bmatrix} = \begin{bmatrix} | & | \\ | & | \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} — \\ — \end{bmatrix}$$

$$A = U \Sigma V^T$$

$$\underbrace{A^T A}_{\text{sym.}} = (U \Sigma V^T)^T U \Sigma V^T$$

$$= V \Sigma^T \underbrace{\cancel{U^T U}}_{I} \Sigma V^T$$

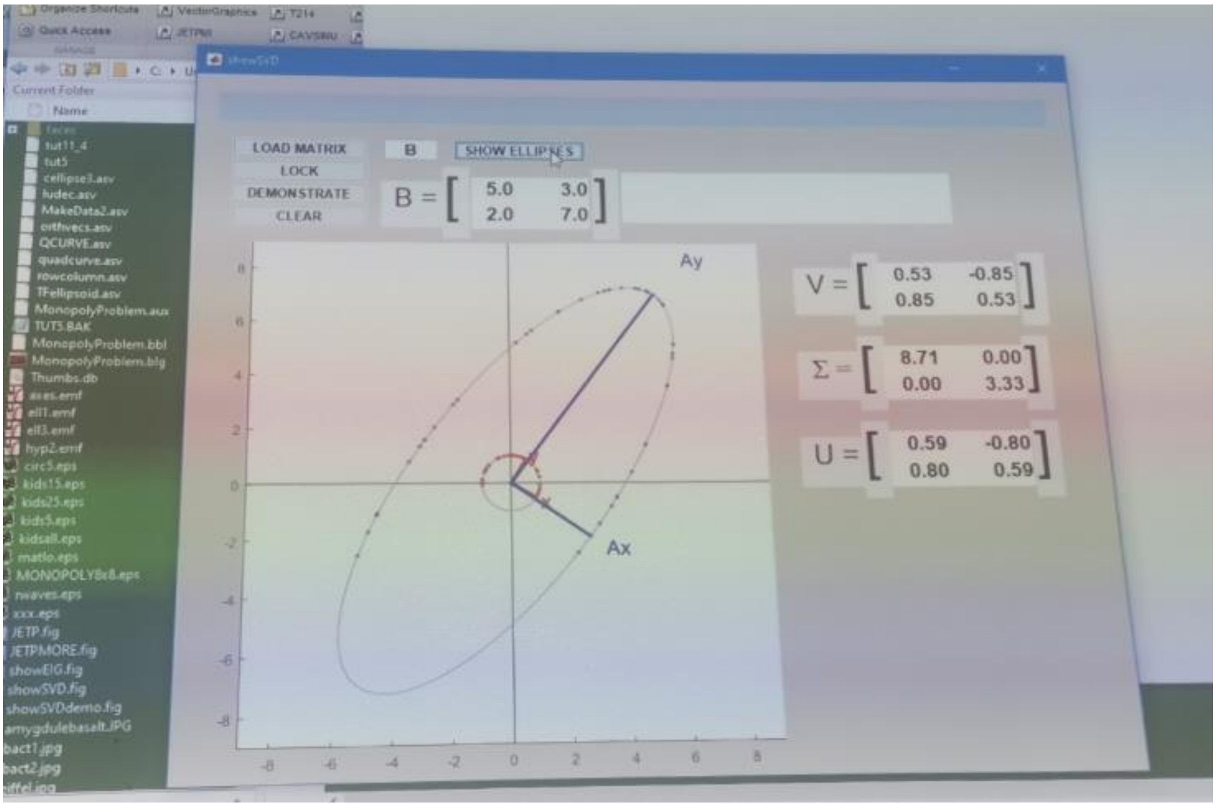
$$= V \underbrace{\Sigma^T \Sigma} V^T$$

$$A^T A = V \Lambda V^T$$

$$\Sigma^T = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

$$\Sigma^T \Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} = \Lambda$$


LOAD MATRIX
LOCK
DEMONSTRATE
CLEAR
B
SHOW ELLIPSES
B = [5.0 3.0 ; 2.0 7.0]
V = [0.53 -0.85 ; 0.85 0.53]
Σ = [8.71 0.00 ; 0.00 3.33]
U = [0.59 -0.80 ; 0.80 0.59]
Ay
Ax