AM214-2023: LECTURE 31

LECTURE 31 — QUADRATIC CURVES $\boxed{1}$

$A = A^T$, can decompose it as $A = Q\Lambda Q^T$.

A is non-degenerate, $\{x_1, x_2, x_3\}$ are automatically orthogonal.

A is degenerate $(\lambda_1, \lambda_1, \lambda_3)$, then

$\{x_1, x_2, x_3\}$ can be chosen orthogonal.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = 5I$$

QUADRATIC CURVE: $\boxed{2}$

$$ax^2 + by^2 + cxy + \cancel{dx} + \cancel{ey} + f = 0$$

$$ax^2 + bxy + cy^2 = d$$

Example: $73x^2 + 72xy + 52y^2 = 100$

$$\underbrace{52}_{aa}y^2 + \underbrace{(72x)}_{bb}y + \underbrace{(73x^2 - 100)}_{cc} = 0$$

$$y = \frac{-bb \pm \sqrt{bb^2 - 4aa.cc}}{2 \cdot aa}.$$

$\boxed{3}$

2×2 ROTATIONS:

$$\underline{i}' = \begin{bmatrix} c_\theta \\ s_\theta \end{bmatrix}, \quad \underline{j}' = \begin{bmatrix} -s_\theta \\ c_\theta \end{bmatrix}$$

$$Q = \begin{bmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{bmatrix}$$

$$Q^T = \begin{bmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{bmatrix}$$

$$Q = \begin{bmatrix} + & - \\ + & + \end{bmatrix} \quad \text{or} \quad Q = \begin{bmatrix} + & + \\ - & + \end{bmatrix}$$

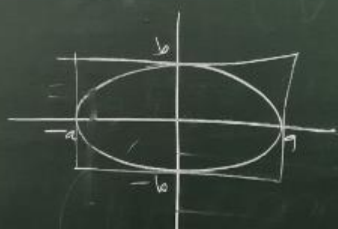
$\boxed{4}$

Quad curves, theory:

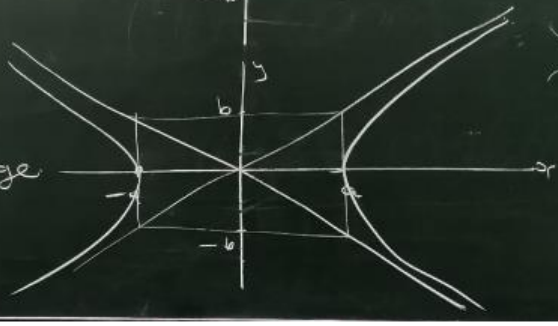
$$ax^2 + bxy + cy^2 = d$$

Conic sections, summary

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



If x large, y large

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} \approx 0$$

$\boxed{5}$

$$\underbrace{\left(\frac{x}{a}-\frac{y}{b}\right)}_{\text{line}}\underbrace{\left(\frac{x}{a}+\frac{y}{b}\right)}_{\text{line}} \approx 0$$

$$-x^2 - y^2 = 1$$

Theory:

$$ax^2 + bxy + cy^2 = d$$

$$\underline{u} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} x & y \end{bmatrix}}_{\underline{u}^T} \underbrace{\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\underline{u}} = d$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} ax + \frac{b}{2}y \\ \frac{b}{2}x + cy \end{bmatrix} = x\left(ax + \frac{b}{2}y\right) + y\left(\frac{b}{2}x + cy\right)$$

$$= ax^2 + bxy + cy^2 = d$$

$\boxed{6}$

$$\underline{u}^T A \underline{u} = d \qquad A = Q \Lambda Q^T$$

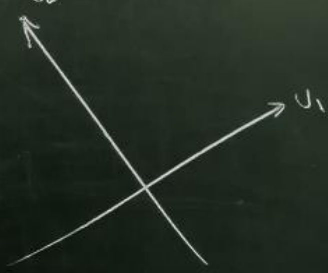
$$\underbrace{\underline{u}^T Q}_{\underline{v}^T} \Lambda \underbrace{Q^T \underline{u}}_{\underline{v}} = d$$

$$\underline{v} = Q^T \underline{u}$$

$$\underline{v}^T = \underline{u}^T Q$$

$$\underline{v}^T \Lambda \underline{v} = d$$

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = d$$



$$\lambda_1 v_1^2 + \lambda_2 v_2^2 = d$$

λ_1, λ_2 both positive

$$\frac{\lambda_1}{d} v_1^2 + \frac{\lambda_2}{d} v_2^2 = 1$$

$\boxed{7}$

$$\frac{V_1^2}{\left(\frac{d}{\lambda_1}\right)} + \frac{V_2^2}{\left(\frac{d}{\lambda_2}\right)} = 1 \qquad a=\sqrt{\frac{d}{\lambda_1}} \qquad b=\sqrt{\frac{d}{\lambda_2}}$$

$$\frac{V_1^2}{\underbrace{\left(\sqrt{\frac{d}{\lambda_1}}\right)^2}_{a}} + \frac{V_2^2}{\underbrace{\left(\sqrt{\frac{d}{\lambda_2}}\right)^2}_{b}} = 1$$

Q must be ~~cost~~ as $\begin{bmatrix} + & \overset{-\sin\theta}{-} \\ + & + \end{bmatrix}$ or $\begin{bmatrix} + & + \\ - & + \end{bmatrix}$

<u>Example</u>: $A=\begin{bmatrix} 73 & 36 \\ 36 & 52 \end{bmatrix}$ $d=100$ $-\sin\theta = 0.8$

$a=\sqrt{\frac{100}{25}}=2,$ $b=\sqrt{\frac{100}{100}}=1$ $\theta = -53.1^\circ$

$\boxed{8}$