## AM214-2023: LECTURE 29

# Lecture 29 — Complex Eigenvalues + 2nd Order Diff'l Eqn's

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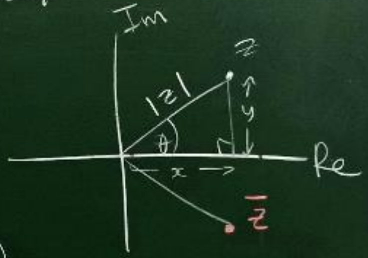
$A \in \mathbb{C}^{m \times n}$

## Review of Complex An.

$i = \sqrt{-1}, \quad i(-i) = 1, \quad \frac{1}{i} = -i, \quad \frac{1}{-i} = i$

$z = x + iy$

$Re(z) = x = \frac{1}{2}(z + \bar{z}) \qquad Im(z) = y = \frac{z - \bar{z}}{2i}$




**Modulus:**

$|z|^2 = \left(\sqrt{x^2 + y^2}\right)^2 = z\bar{z}$

**Argument:**

$\theta = \arctan\left(\frac{y}{x}\right)$

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**Conjugate:**

$z = x + iy$

$\bar{z} = x - iy$

$z + \bar{z}$ is real

$z\bar{z}$ is real

$z + \bar{z} = x + iy + x - iy = 2x$

$z - \bar{z} = x + iy - (x - iy) = +2iy$

$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$

$f(z) = \frac{e^{iz} + \cos(z)}{z + 5} \xrightarrow{\text{split}} Re + i\, Im.$

$f(z) = \frac{z + 5i}{3z + 4} \cdot \frac{3\bar{z} + 4}{3\bar{z} + 4}$

$\boxed{3}$

$$= \frac{(z+5i)(3\bar{z}+4)}{(3z+4)(3\bar{z}+4)} = \frac{3z\bar{z} + 4z + 15i\bar{z} + 20i}{9z\bar{z} + 12z + 12\bar{z} + 16}$$

$$= \frac{\qquad\qquad}{\underset{\substack{\uparrow \\ \text{real}}}{9z\bar{z}} + 12\underset{\substack{\uparrow \\ \text{real}}}{(z+\bar{z})} + 16}$$

Euler's formula

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

Taylor: $e^{\theta} = 1 + \theta + \frac{1}{2!}\theta^2 + \frac{1}{3!}\theta^3 + \cdots$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots$$

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$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)$$

$$= \cos x + i\sin x$$

$$\boxed{\begin{aligned} e^{ix} &= \cos x + i\sin x \\ e^{-ix} &= \cos x - i\sin x \end{aligned}}$$

$$\boxed{\begin{aligned} \cos x &= \frac{e^{ix}+e^{-ix}}{2} \\ \sin x &= \frac{e^{ix}-e^{-ix}}{2i} \end{aligned}}$$

$$\frac{e^{ix}+e^{-ix}}{2} = \cos x$$

$$\frac{e^{ix}-e^{-ix}}{2i} = \cancel{2}i\sin x$$

## Complex Matrices $\boxed{5}$

$\mathbb{C}^{m\times n}$

### Transpose:

$$\underline{a}^T\underline{a} = a_1^2 + a_2^2 + \dots a_n^2 = \|\underline{a}\|^2$$

$$\underline{z} = \begin{bmatrix} a+ib \\ c+id \end{bmatrix} \qquad \underline{\text{Try:}} \quad \begin{bmatrix} a+ib & c+id \end{bmatrix}\begin{bmatrix} a+ib \\ c+id \end{bmatrix}$$

$$= (a+ib)(a+ib) + (c+id)(c+id)$$

$$= \underbrace{a^2 - b^2 + 2iab + c^2 - d^2 + 2icd}_{\text{not real, not positive}}$$

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### Try: $\boxed{6}$

$$\begin{bmatrix} a-ib & c-id \end{bmatrix}\begin{bmatrix} a+ib \\ c+id \end{bmatrix}$$

$$= (a-ib)(a+ib) + (c-id)(c+id)$$

$$= \underbrace{a^2+b^2+c^2+d^2}_{\text{real, positive}}, \quad = 0 \text{ if } \underline{z} = \underline{0}$$

$$\cancel{(\bar{\underline{z}})^T\underline{z} = \|\underline{z}\|^2}$$

$$\underline{a} \in \mathbb{C}^n, \quad \underline{a}^T = \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_n \end{bmatrix}$$

Hermitian transpose $\quad \underline{a}^H \quad \underline{a}^T$

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$$A = \begin{bmatrix} a & b & x \\ c & d & y \end{bmatrix} \quad a, b, c, d \in \mathbb{C}$$

$$A^T = \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \\ \bar{x} & \bar{y} \end{bmatrix}$$

Example: $X, A, B, D \in \mathbb{C}^{?\times ?}$, $\alpha \in \mathbb{C}$

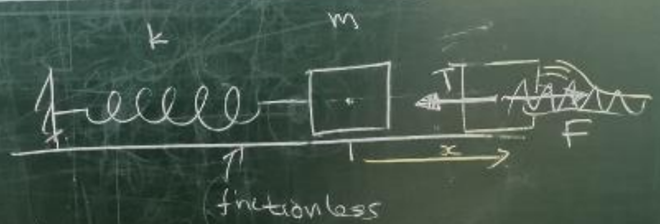
$$X = (\alpha A + BC + D^T)^T$$

$$= \bar{\alpha} A^T + C^T B^T + D$$

$\boxed{8}$

# $2^{ND}$ ORDER DIFF'L EQN'S

SHM




$T = -kx$

$ma = F$

$m\ddot{x} = -kx$

$$\ddot{x} + \underbrace{\frac{k}{m}}_{\omega^2} x = 0$$

Let $\omega^2 = \frac{k}{m}$

$$[\omega^2] = \frac{N.m^{-1}}{kg}$$

$$= \frac{\cancel{kg}.\cancel{m}.s^{-2}.\cancel{m^{-1}}}{\cancel{kg}} = s^{-2}$$

$[\omega] = s^{-1}$
$= Hz.$

Turn into 1st order system

$$\ddot{x} = -\omega^2 x$$

Let $v = \dot{x}$, $\dot{v} = \ddot{x}$

$$\left.\begin{aligned} \dot{v} &= -\omega^2 x \\ \dot{x} &= v \end{aligned}\right\} \longrightarrow \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

$$x(0) = x_0$$
$$v(0) = v_0$$

$$\lambda^2 - 0\lambda + \omega^2 = 0$$

$$\lambda^2 = -\omega^2 \Rightarrow \lambda = \pm\sqrt{-\omega^2}$$

$$\lambda_1 = i\omega, \quad \lambda_2 = -i\omega$$

$\underline{\lambda_1 = i\omega}$



$$\begin{bmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{bmatrix} \xrightarrow{LU} \begin{bmatrix} -\lambda & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \mu \end{bmatrix} = \underline{0}$$

$$-\lambda x_1 + \mu = 0$$

$$\lambda x_1 = \mu$$

$$x_1 = \frac{\mu}{\lambda}$$

$$\underline{x} = \begin{bmatrix} \frac{\cancel{\mu}}{\lambda} \\ \cancel{\mu} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix} \quad S = \begin{bmatrix} 1 & 1 \\ i\omega & -i\omega \end{bmatrix}, \quad S^{-1} = \frac{1}{-2i\omega} \begin{bmatrix} -i\omega & -1 \\ -i\omega & 1 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ v(t) \end{bmatrix} = \frac{1}{-2i\omega} \begin{bmatrix} 1 & 1 \\ i\omega & -i\omega \end{bmatrix} \begin{bmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{bmatrix} \begin{bmatrix} -i\omega & -1 \\ -i\omega & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$