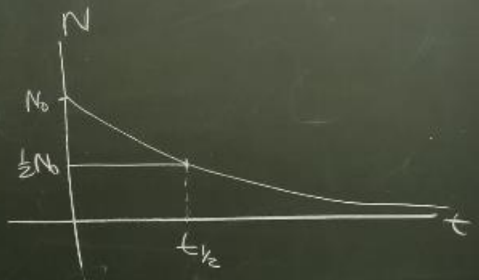
AM214-2023: LECTURE 28

$\boxed{\text{LECTURE 28}}$ SYSTEMS OF DIFFERENTIAL EQN'S $\boxed{1}$

Radio-active decay chains

$N(t)$ = number of atoms.

~~dif'l eq~~ $$\frac{dN}{dt} = -\alpha N$$



~~Sol~~ $$N(t) = N_0 e^{-\alpha t}$$

$$\frac{1}{2}\cancel{N_0} = \cancel{N_0} e^{-\alpha t_{1/2}}$$

$$\ln 2 = +\alpha t_{1/2}$$

$$t_{1/2} = \frac{\ln 2}{\alpha}$$

$\boxed{2}$

Example:

$A \longrightarrow B$ 0.15 per day

$B \rightarrow C$ 0.20 per day

$C \rightarrow$ stable.

Initially

80% A

20% B

Let $a(t)$ be the % of atoms of A.

$b(t)$ % of " B

$c(t)$ C

$$u(0) = \begin{bmatrix} 80\% \\ 20\% \end{bmatrix}$$

$$\dot{a} = -0.15\, a$$

$$\dot{b} = 0.15\, a - 0.2\, b$$

$$\cancel{\dot{c} = 0.2\, b}$$

$$\underbrace{\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix}}_{\dot{u}} = \underbrace{\begin{bmatrix} -0.15 & 0 \\ 0.15 & -0.20 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{u}$$

$\boxed{3}$

$$\dot{\underline{y}} = A\underline{y} \qquad \underline{y}(0) = \underline{y}_0$$

$$\dot{\underline{y}} = S\Lambda S^{-1}\underline{y}$$

$$\underbrace{S^{-1}\dot{\underline{y}}}_{\dot{\underline{w}}} = \Lambda \underbrace{S^{-1}\underline{y}}_{\underline{w}} \qquad \boxed{\underline{y} = S\underline{w}}$$

$$\dot{\underline{w}} = \Lambda \underline{w}$$

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\dot{w}_1 = \lambda_1 w_1 \qquad w_1(t) = w_1(0)e^{\lambda_1 t}$$

$$\dot{w}_2 = \lambda_2 w_2 \qquad w_2(t) = w_2(0)e^{\lambda_2 t}$$

$$\underline{w} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \underline{w}(0)$$

$$\overset{\underline{y}}{\cancel{S\underline{w}}} = S\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} S^{-1}\underline{y}(0)$$

$$\boxed{\underline{y}(t) = S e^{t\Lambda} S^{-1} \underline{y}(0)}$$

$\boxed{4}$

Example:

$$\dot{x} = -5x + 6y \qquad x(0) = 4$$

$$\dot{y} = -x \qquad y(0) = 20$$

$$\dot{\underline{y}} = \begin{bmatrix} -5 & 6 \\ -1 & 0 \end{bmatrix} \underline{y} \qquad \underline{y}(0) = \begin{bmatrix} 4 \\ 20 \end{bmatrix}$$

$$\lambda^2 + 5\lambda + 6 = 0$$

$$(\lambda+3)(\lambda+2) = 0$$

$$\lambda_1 = -3, \quad \lambda_2 = -2$$

$\boxed{\lambda_1 = -3}$

$$\begin{bmatrix} -2 & 6 \\ -1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \mu \end{bmatrix} = \underline{0}$$

$$-x_1 + 3\mu = 0, \quad x_1 = 3\mu$$

$$\underline{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\boxed{5}$

$\boxed{\lambda_2 = -2}$

$$\begin{bmatrix} -3 & 6 \\ -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & 6 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{0} \qquad -x_1 + 2x_2 = 0$$

$$\underline{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}, \quad S^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

Solution:

$$\underline{y}(t) = S e^{t\Lambda} S^{-1} \underline{y}(0)$$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{-3t} & 2e^{-2t} \\ e^{-3t} & e^{-2t} \end{bmatrix} \begin{bmatrix} -36 \\ 56 \end{bmatrix}$$

$\boxed{6}$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -108\,e^{-3t} + 112\,e^{-2t} \\ -36\,e^{-3t} + 56\,e^{-2t} \end{bmatrix}$$

Find time when x is maximum:

$$\dot{x} = (-3)(-108)e^{-3t} - 2(112)e^{-2t} = 0$$

$$\times e^{3t}: \quad 324\, e^{0} - 224\, \overset{e^{t}}{\cancel{e^{-2t}\, e^{3t}}} = 0$$

$$e^{t} = \frac{324}{224}$$

$$t = \ln\left(\frac{324}{224}\right)$$

$\boxed{7}$

Example in Prob-set 13.7

Second order linear differential eqn's.

$$\ddot{x} + a\dot{x} + bx = 0 \qquad \boxed{\text{Let } x = e^{rt}}$$

Rewrite as 2×2 first order system.

Let $\dot{x} = v$, $\ddot{x} = \dot{v}$

$$\boxed{\begin{aligned} &\dot{v} + av + bx = 0 \\ &\dot{x} = v \end{aligned}}$$

$\boxed{8}$

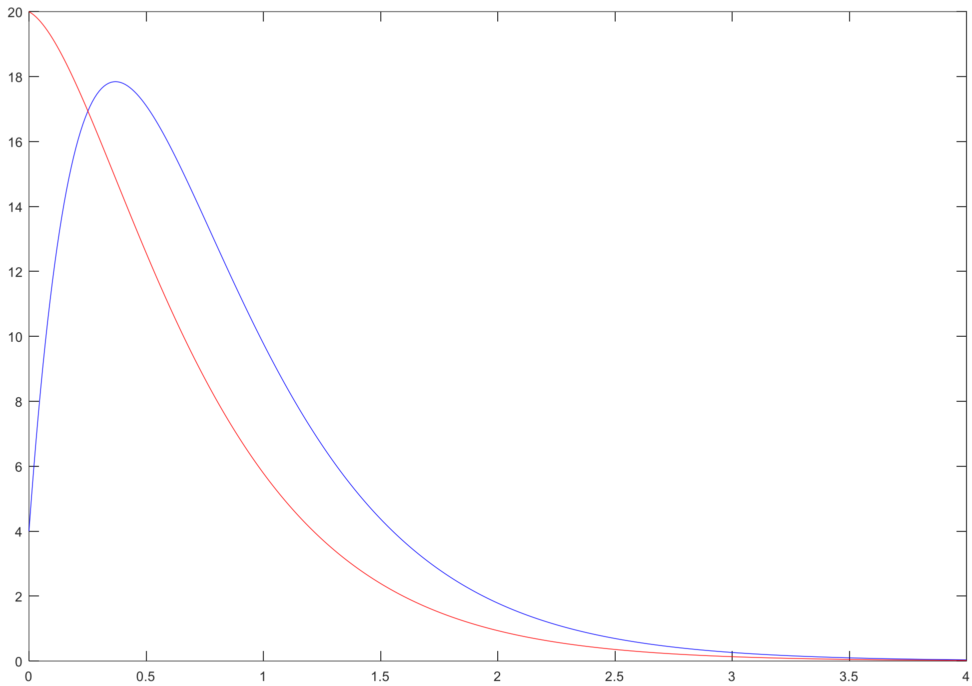
$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -bx - av \end{aligned} \qquad \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

$$\cancel{\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -a & -b \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix}}$$

Do not do this!

$P\underline{u}$

$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$


20
18
16
14
12
10
8
6
4
2
0
0
0.5
1
1.5
2
2.5
3
3.5
4