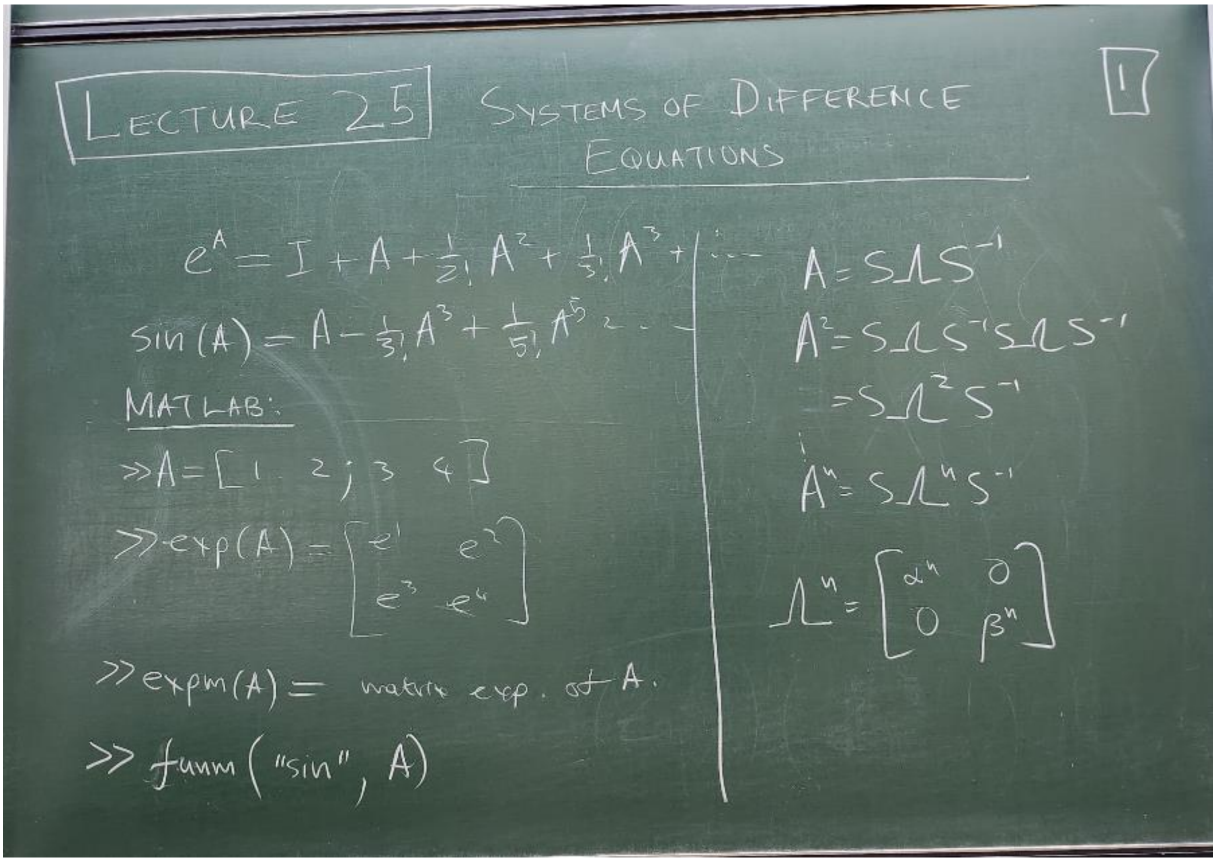
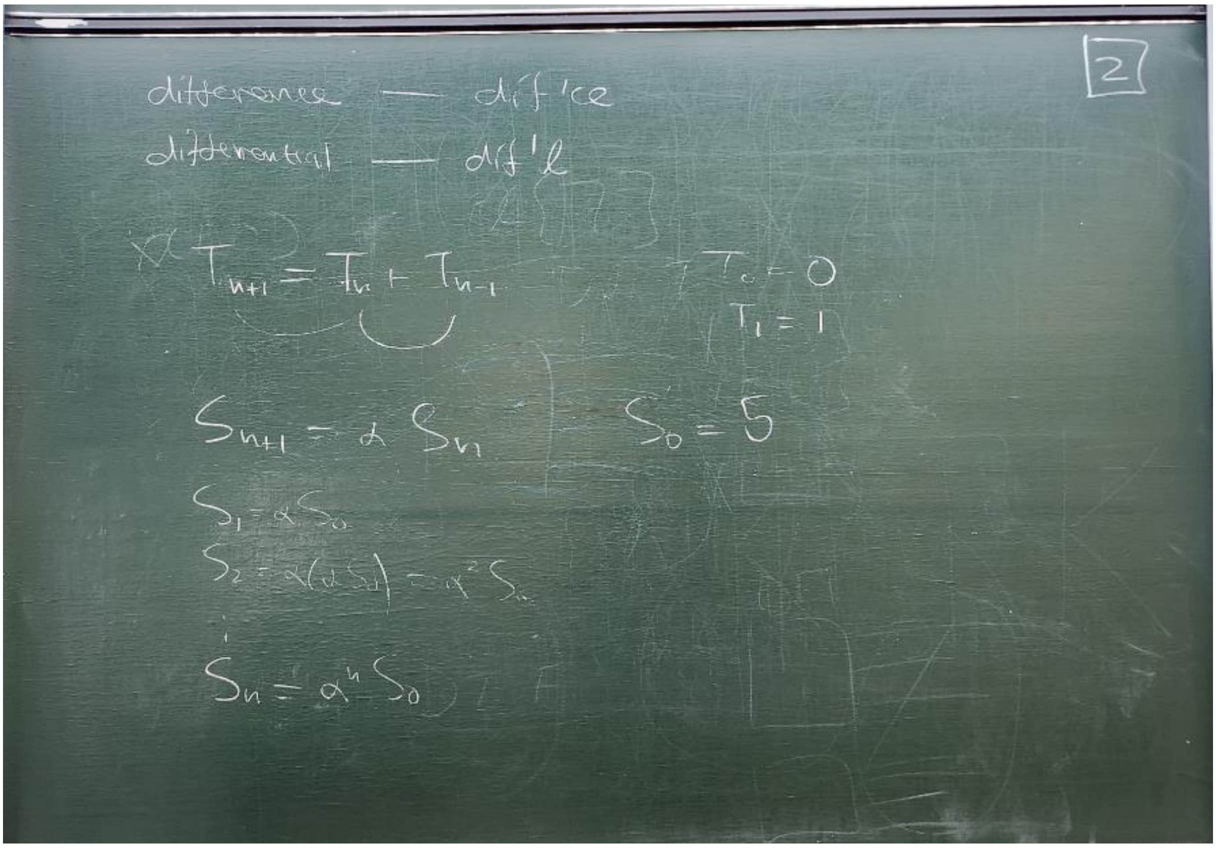
AM214-2023: LECTURE 25





3

SYSTEM OF 2 1'ST ORDER DIFF'CE EQN'S

$$x_{n+1} = a x_n + b y_n \qquad x_0 = \text{given}$$
$$y_{n+1} = c x_n + d y_n \qquad y_0 = \text{given}$$

Rent-Sum-Wheels:

		Initially
Let J_n be the # cars in	JNB	250
C_n	CPT	90
D_n	DBN.	60

$J \xrightarrow{20\%} C$, $J \xrightarrow{5\%} D$

$C \xrightarrow{15\%} J$, $C \xrightarrow{45\%} D$

$D \xrightarrow{35\%} J$, $D \xrightarrow{20\%} C$.

4

$$J_{n+1} = 0.75 J_n + 0.15 C_n + 0.35 D_n$$
$$C_{n+1} =$$

$$x_{n+1} = a x_n + b y_n. \qquad x_0 = \text{given}$$
$$y_{n+1} = c x_n + d y_n \qquad y_0 = \text{given}.$$

$$\underbrace{\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix}}_{\underline{u}_{n+1}} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_n \\ y_n \end{bmatrix}}_{\underline{u}_n} \qquad \underline{u}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

SYS. OF DIFF'CE: $\boxed{\underline{u}_{n+1} = A \underline{u}_n}$

INITIAL COND.: $\boxed{\underline{u}_0 \text{ given.}}$

5

$\underline{y}_1 = A\underline{y}_0$

$\underline{y}_2 = A(A\underline{y}_0) = A^2\underline{y}_0$

$\vdots$

$\boxed{\underline{y}_k = A^k\underline{y}_0}$ ← SOLUTION.

Decompose A as $A = S\Lambda S^{-1}$

$A^k = S\Lambda^k S^{-1}$

$$\boxed{\underline{y}_k = S\Lambda^k S^{-1}\underline{y}_0}$$

6

Example: Markov chain

States A and B.

A flips to B with probability 0.4

B flips to A " 0.2

Initially 700 in state A, 300 in state B.

Let a_n be number of particles in state A at time n.

b_n " B "

$a_{n+1} = 0.6a_n + 0.2b_n$

$b_{n+1} = 0.4a_n + 0.8b_n$

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 700 \\ 300 \end{bmatrix}$$

7

$$\underline{y}_{n+1} = A\underline{y}_n \qquad A = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$$

Markov matrix:

- columns sum to 1.
- components are ≤ 1 and ≥ 0

Solution:

$$\underline{y}_k = S\Lambda^k S^{-1}\underline{y}_0$$

Eig-decomp of A:

$$\begin{vmatrix} 0.6-\lambda & 0.2 \\ 0.4 & 0.8-\lambda \end{vmatrix} = 0 \qquad \lambda^2 - 1.4\lambda + 0.4 = 0$$

8

$$(1-\lambda)(0.4-\lambda) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = 0.4$$

$\lambda = 1$

$$\begin{bmatrix} -0.4 & 0.2 \\ 0.4 & -0.2 \end{bmatrix} \xrightarrow{LU} \begin{bmatrix} -0.4 & 0.2 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-0.4x_1 + 0.2\mu = 0 \qquad x_1 = \tfrac{1}{2}\mu$$

$$\underline{x}_1 = \begin{bmatrix} \tfrac{1}{2}\mu \\ \mu \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\lambda_2 = 0.4$

$$\begin{bmatrix} 0.2 & 0.2 \\ 0.4 & 0.4 \end{bmatrix} \rightarrow \begin{bmatrix} 0.2 & 0.2 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ \mu \end{pmatrix} = \underline{0}$$

$$x_1 = -\mu$$

$$\underline{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix},$$

$\boxed{9}$

$$S = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad S^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\underline{u}_k = \frac{1}{3}\underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & (0.4)^k \end{bmatrix}}\underbrace{\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 700 \\ 300 \end{bmatrix}}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & -(0.4)^k \\ 2 & (0.4)^k \end{bmatrix}\begin{bmatrix} 1000 \\ -1100 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1000 + (0.4)^k\, 1100 \\ 2000 - (0.4)^k\, 1100 \end{bmatrix}$$

$$\boxed{\begin{aligned} x_k &= (1000 + 1100(0.4)^k)/3 \\ y_k &= (2000 - (0.4)^k\, 1100)/3 \end{aligned}}$$

$x_0 = 700$

$y_0 = 300$