AM214-2023: LECTURE 24

$\boxed{\text{LECTURE 24}}$ EIGENVALUES, PROPERTIES

$\boxed{1}$

A is square

$|A-\lambda I| = 0$ ← characteristic equation.

roots are eigenvalues.

$\lambda_1 \quad \underline{x}_1$

$\lambda_2 \quad \underline{x}_2$

$\lambda_3 \quad \underline{x}_3$

$\boxed{2\times 2}$

$$\left|\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right| = \lambda^2 - tr(A)\lambda + det(A) \text{ — expanded form}$$

$$= (\alpha-\lambda)(\beta-\lambda) \text{ ← factored form.}$$

$\boxed{2}$

$$= \lambda^2 - (\alpha+\beta)\lambda + \alpha\beta$$

$tr(A) = \alpha+\beta$ = sum eval's

$det(A) = \alpha\beta$ = prod. eval's

$\boxed{3\times 3}$

$$\begin{vmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{vmatrix} \begin{matrix} a-\lambda & b \\ d & e-\lambda \\ g & h. \end{matrix}$$

$$= -cg(e-\lambda) - fh(a-\lambda) - bd(i-\lambda) + (a-\lambda)(e-\lambda)(i-\lambda)$$

$$+ bfg + cdh.$$

$i\lambda^2 + a\lambda^2 + e\lambda^2$

do it:

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$$= -\lambda^3 + (a+e+i)\lambda^2 - (\ldots\ldots\ldots)\lambda$$

$$+ \underbrace{aei + bfg + cdh - cge - ahf - bdi}_{\det(A)}$$

$$= -\lambda^3 + \mathrm{tr}(A)\lambda^2 - h(A)\lambda + \det(A)$$

$$h(A) = ae + ei + ia - bd - cg - fh.$$

Let eigenvalues be α, β, γ.

$$= (\alpha-\lambda)(\beta-\lambda)(\gamma-\lambda)$$

$$= (\alpha\beta - \alpha\lambda - \beta\lambda + \lambda^2)(\gamma-\lambda)$$

$$= -\lambda^3 + (\alpha+\beta+\gamma)\lambda^2 - (\alpha\beta + \beta\gamma + \gamma\alpha)\lambda + \alpha\beta\gamma$$

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$$\boxed{\begin{array}{l} \oplus\ \mathrm{tr}(A) = \alpha+\beta+\gamma = \sum \text{eval's} \\ \otimes\ \det(A) = \alpha\beta\gamma = \prod \text{eval's} \end{array}}$$

Example:

$$A = \begin{bmatrix} 7 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 7 \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$A\underline{x} = \begin{bmatrix} 6 \\ 0 \\ -6 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Let eval's be $\alpha, \beta, 6$

$\oplus$ $\alpha + \beta + 6 = 20 \longrightarrow \alpha + \beta = 14$

$\otimes$ $\alpha\beta 6 = 240 \longrightarrow \alpha\beta = 40$

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$$\boxed{\lambda_1 = 10,\ \lambda_2 = 6,\ \lambda_3 = 4}$$

$$\beta = \frac{40}{\alpha}$$

$$\alpha + \frac{40}{\alpha} - 14 = 0$$

$$\alpha^2 - 14\alpha + 40 = 0$$

$$\alpha = \frac{+14 \pm \sqrt{14^2 - 4(40)}}{2}$$

$$= 4, \text{ or } 10.$$

$$\begin{bmatrix} -3 & -2 & 1 \\ -2 & -4 & -2 \\ 1 & -2 & -3 \end{bmatrix} \to \begin{bmatrix} -3 & -2 & 1 \\ & & \\ & & \end{bmatrix}$$

$\lambda_1 = 10$

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 6$ $\quad x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\lambda_3 = 4,$ $\quad x_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

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$$S = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A = S \Lambda S^{-1}$$

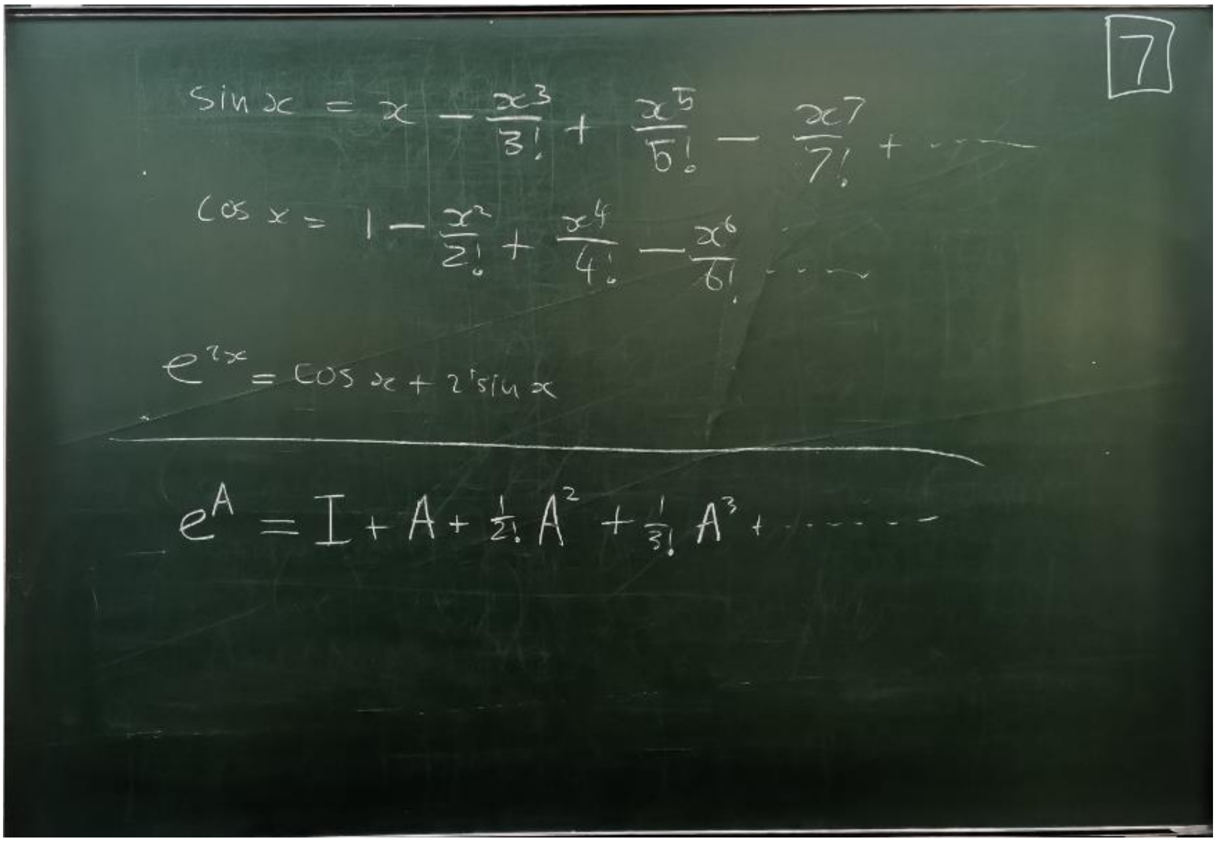
<u>Functions of matrices.</u> CORDIC.

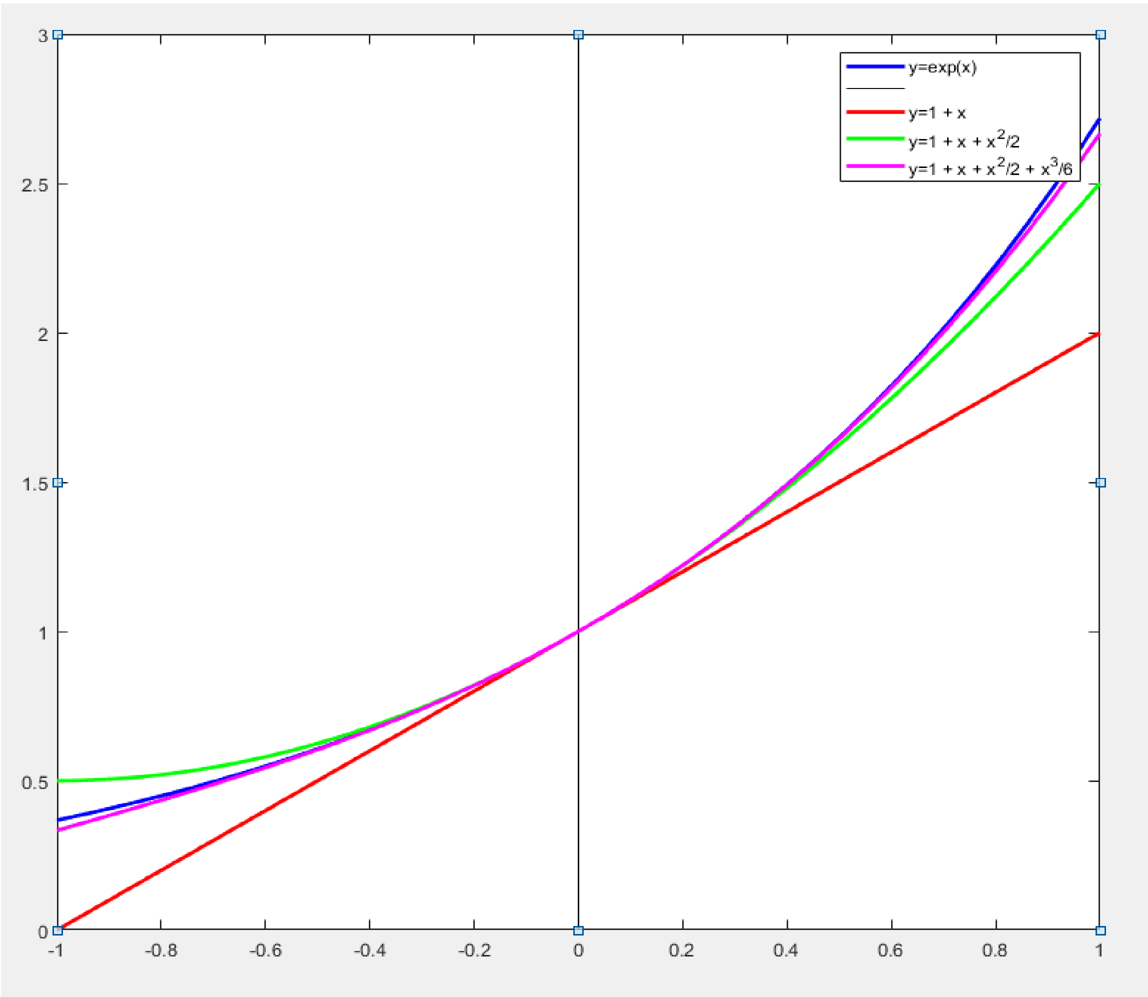
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Check:

$$\frac{d}{dx}(e^x) = 1 + \frac{\cancel{2}x}{\cancel{2}\cdot 1} + \frac{\cancel{3}x^2}{\cancel{3}\cdot 2 \cdot 1} + \frac{\cancel{4}x^3}{\cancel{4}\cdot 3 \cdot 2 \cdot 1} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$$


7
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$
$e^{ix} = \cos x + i \sin x$
$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$


y=exp(x)
y=1 + x
y=1 + x + x^2/2
y=1 + x + x^2/2 + x^3/6
0
0.5
1
1.5
2
2.5
3
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1