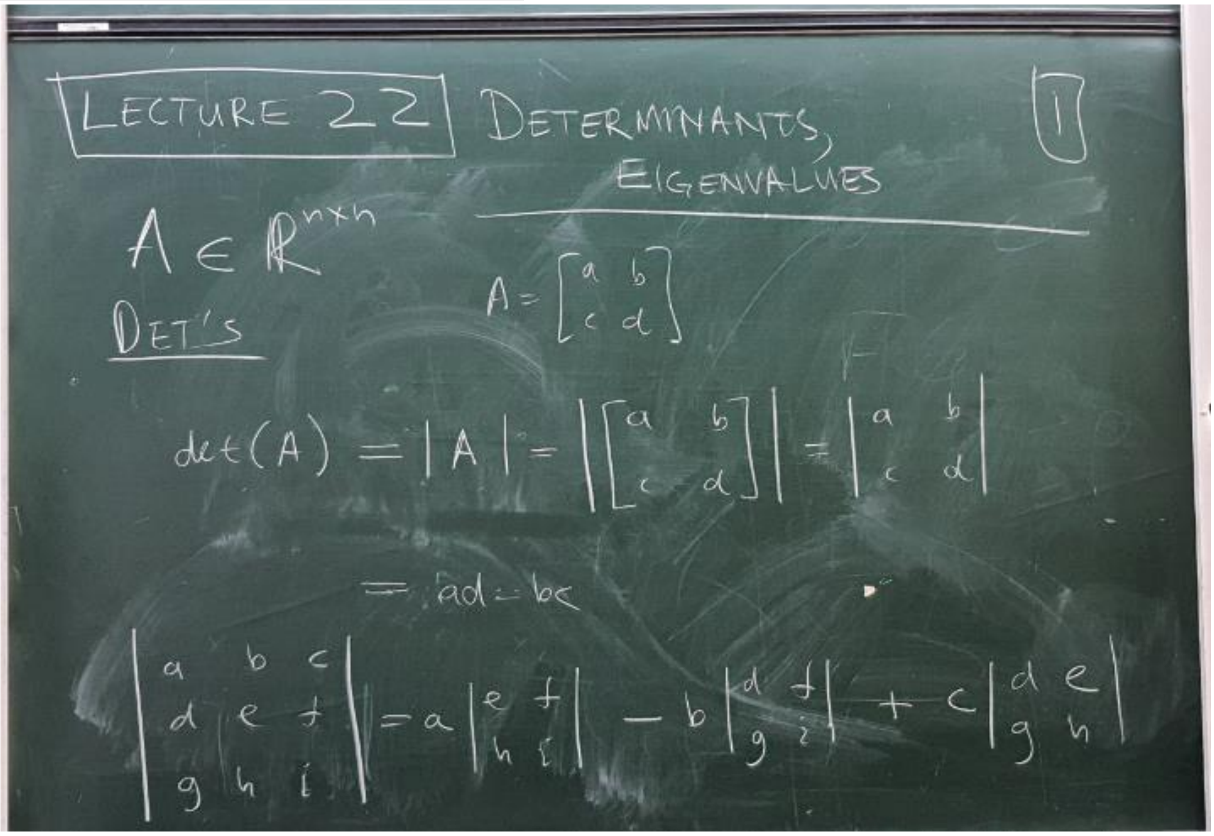
AM214-2023: LECTURE 22



$\boxed{\text{LECTURE 22}}$ DETERMINANTS, EIGENVALUES $\boxed{1}$

$A \in \mathbb{R}^{n\times n}$

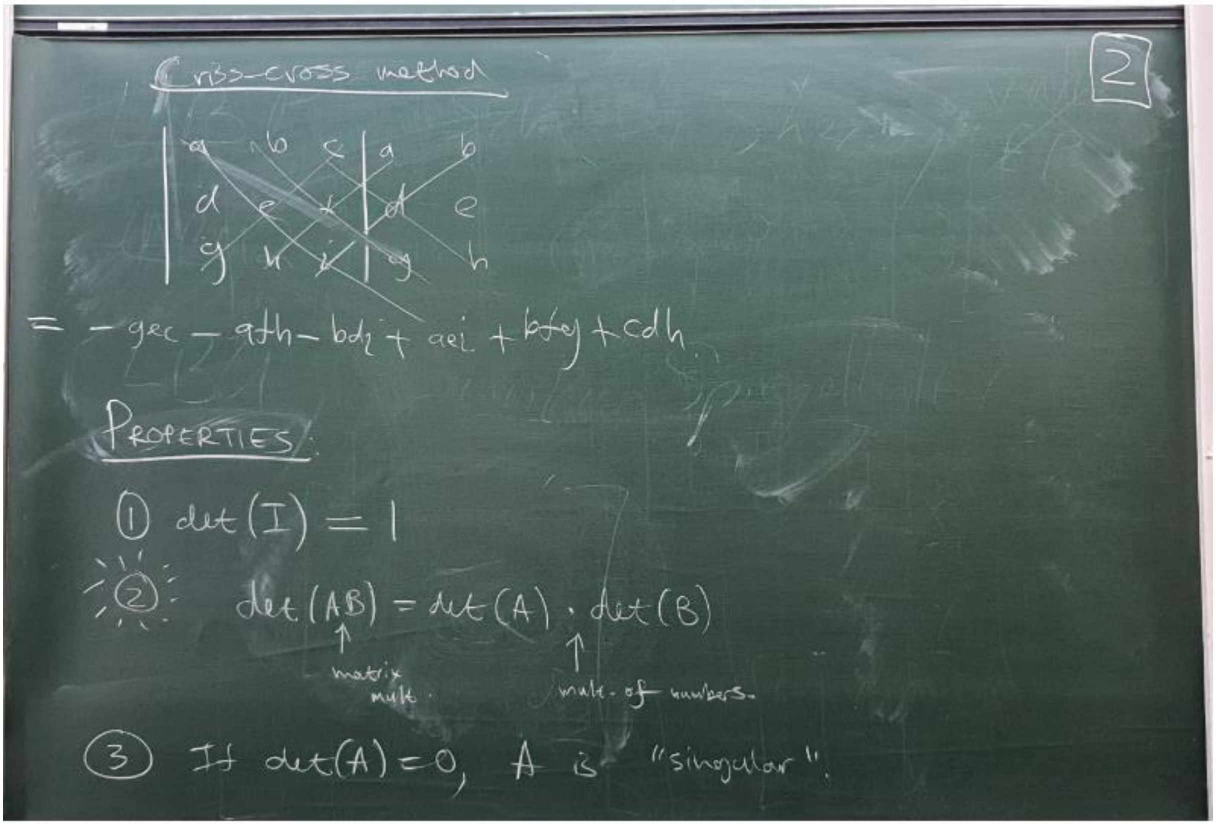
DET'S

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det(A) = |A| = \left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$



Criss-cross method $\boxed{2}$

$$\left|\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right| \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

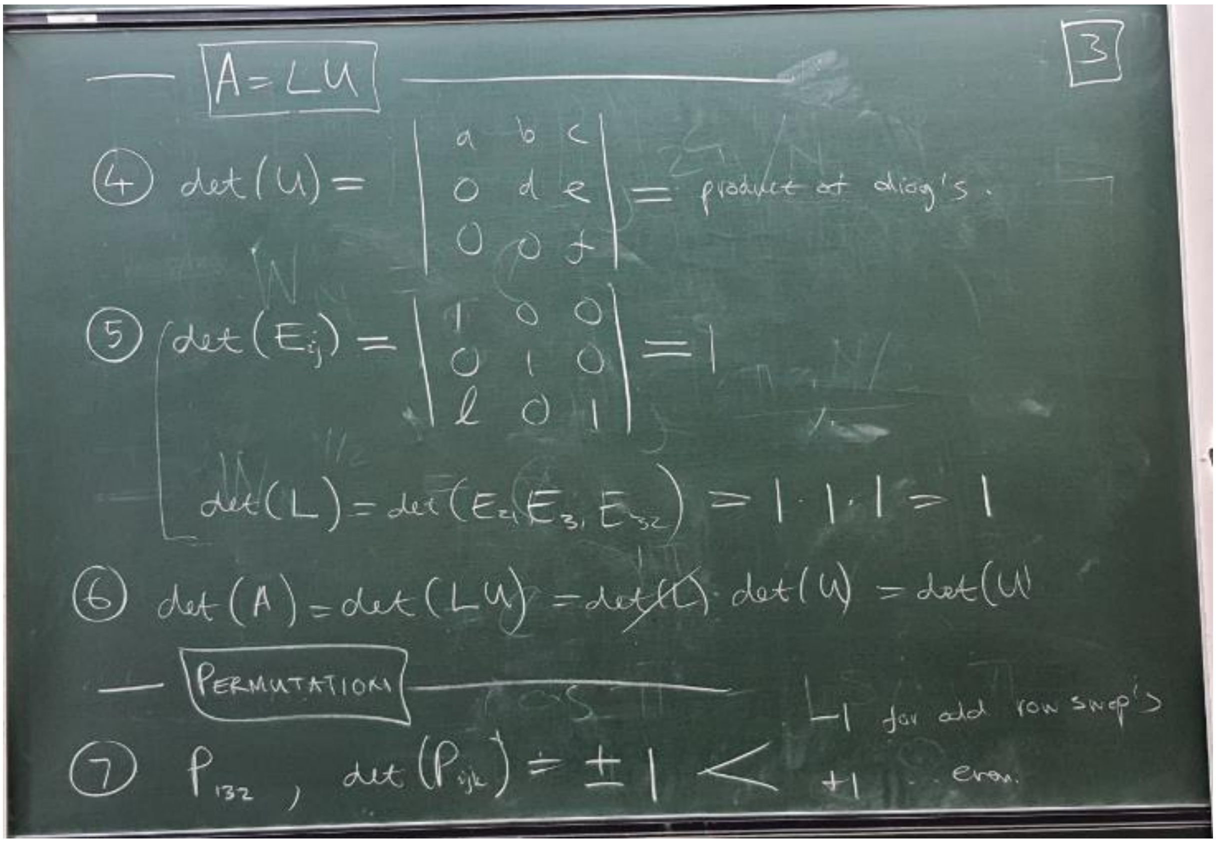
$$= -gec - afh - bdi + aei + bfg + cdh.$$

PROPERTIES:

① $\det(I) = 1$

② $\det(\underset{\substack{\uparrow \\ \text{matrix mult.}}}{AB}) = \det(A) \underset{\substack{\uparrow \\ \text{mult. of numbers.}}}{\cdot} \det(B)$

③ If $\det(A) = 0$, A is "singular".



$\boxed{3}$

A=LU

④ $\det(U) = \begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix}$ = product of diag's.

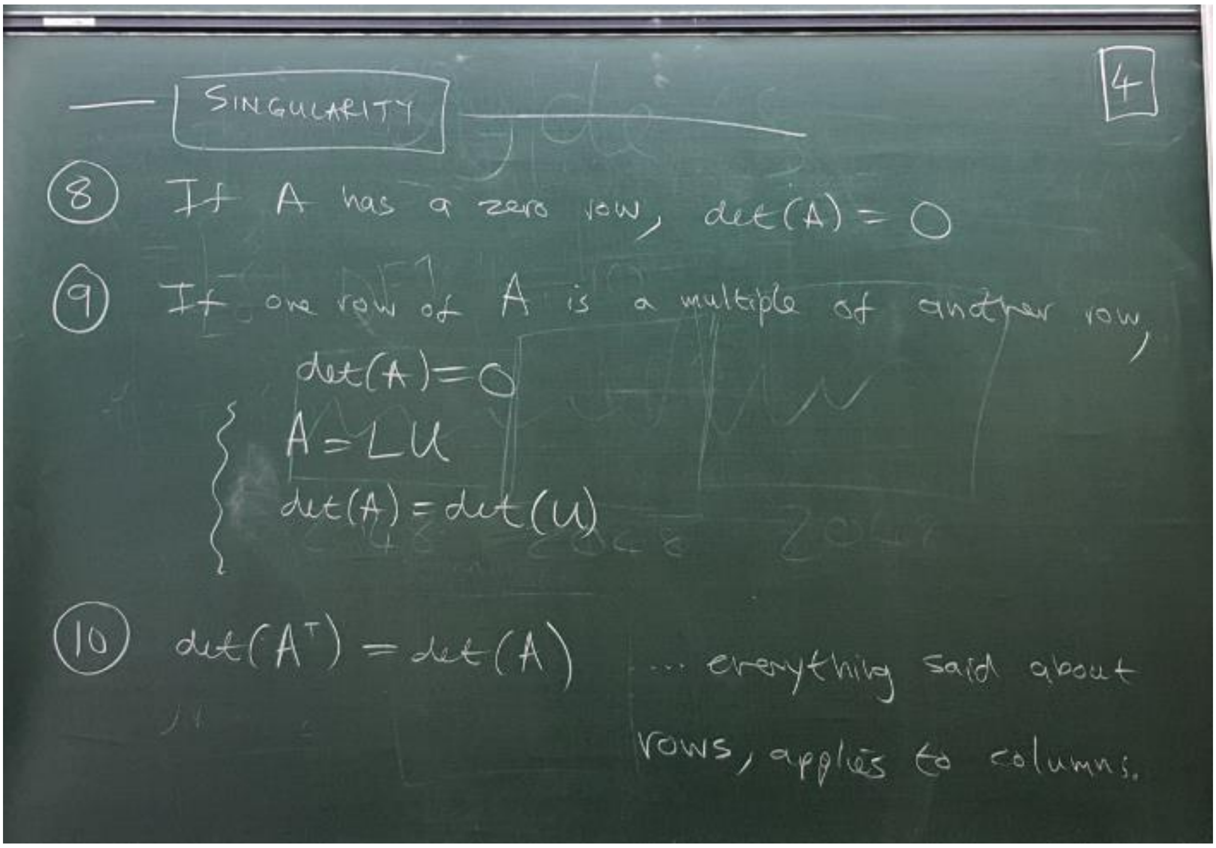
⑤ $\det(E_{ij}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l & 0 & 1 \end{vmatrix} = 1$

$\det(L) = \det(E_{21} E_{31} E_{32}) = 1 \cdot 1 \cdot 1 = 1$

⑥ $\det(A) = \det(LU) = \cancel{\det(L)} \cdot \det(U) = \det(U)$

PERMUTATION

⑦ P_{132}, $\det(P_{ijk}) = \pm 1 <$ -1 for odd row swap's, $+1$ even.



$\boxed{4}$

SINGULARITY

⑧ If A has a zero row, $\det(A) = 0$

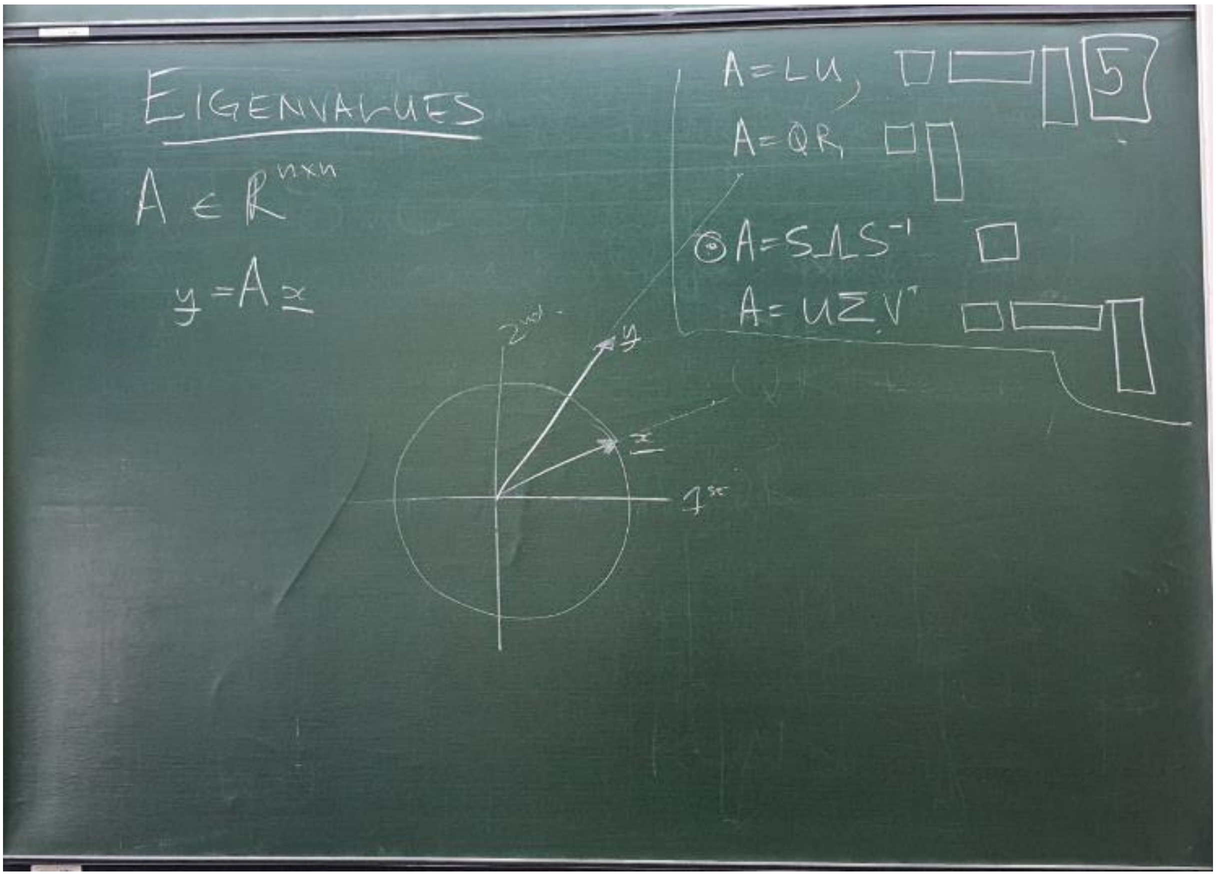
⑨ If one row of A is a multiple of another row,

$\det(A) = 0$

$A = LU$

$\det(A) = \det(U)$

⑩ $\det(A^T) = \det(A)$ … everything said about rows, applies to columns.



EIGENVALUES

$\boxed{5}$

$A \in \mathbb{R}^{n \times n}$

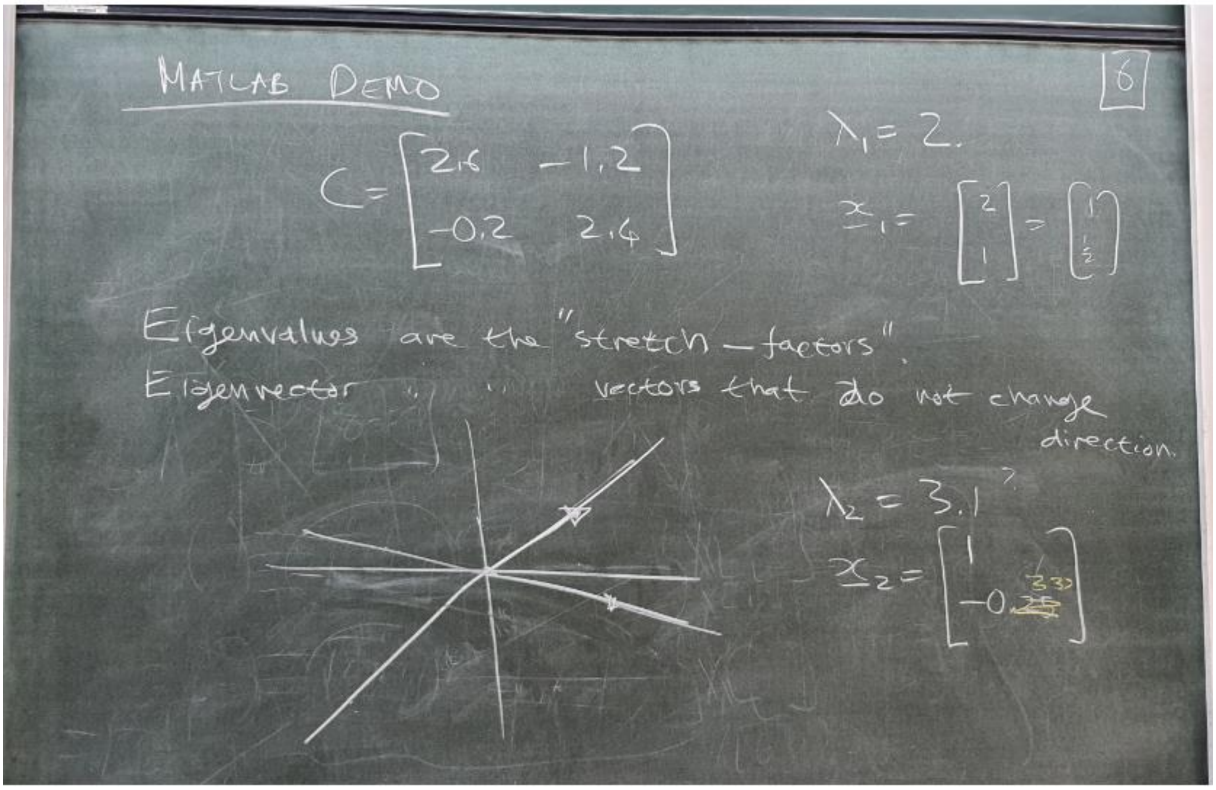
$\underline{y} = A\underline{x}$

$A = LU$, $\square\,\square\,\square$

$A = QR$, $\square\,\square$

$\odot\, A = S \Lambda S^{-1}$ $\square$

$A = U \Sigma V^T$ $\square\,\square\,\square$



MATLAB DEMO

$\boxed{6}$

$$C = \begin{bmatrix} 2.6 & -1.2 \\ -0.2 & 2.4 \end{bmatrix}$$

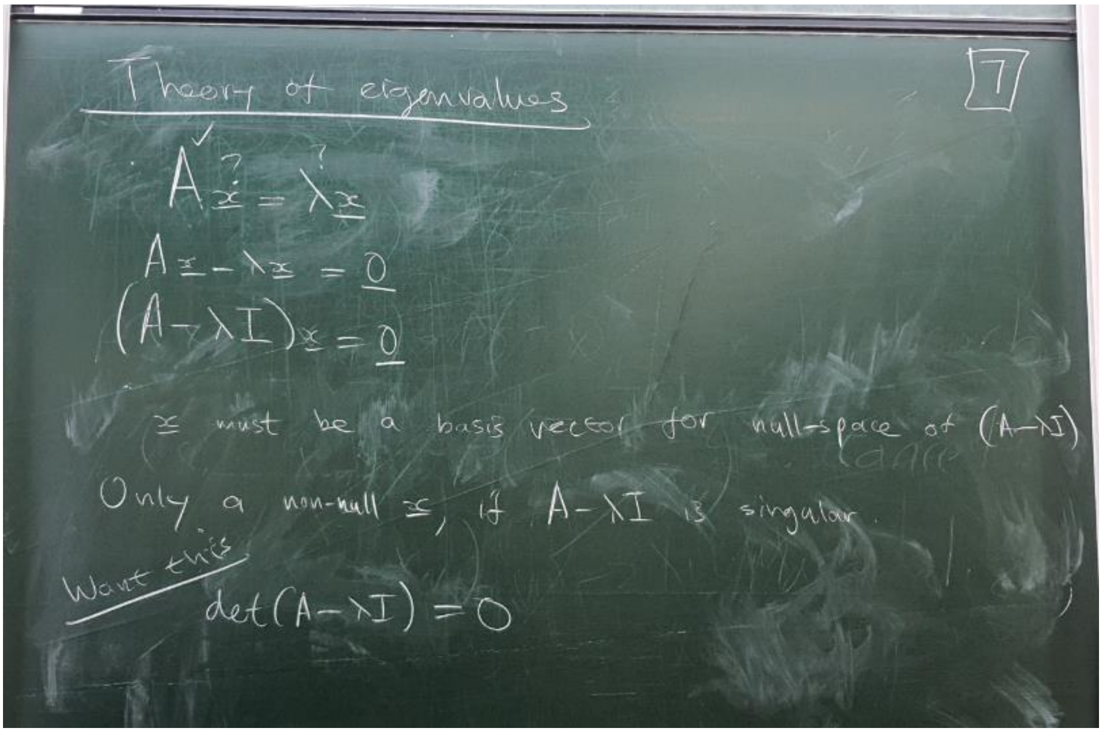
$\lambda_1 = 2.$

$$\underline{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

Eigenvalues are the "stretch – factors".
Eigenvector " " vectors that do not change direction.

$\lambda_2 = 3.1$?

$$\underline{x}_2 = \begin{bmatrix} 1 \\ -0.\overset{333}{\cancel{25}} \end{bmatrix}$$



Theory of eigenvalues

$\boxed{7}$

$$\overset{\checkmark}{A}\overset{?}{\underline{x}} = \overset{?}{\lambda}\underline{x}$$

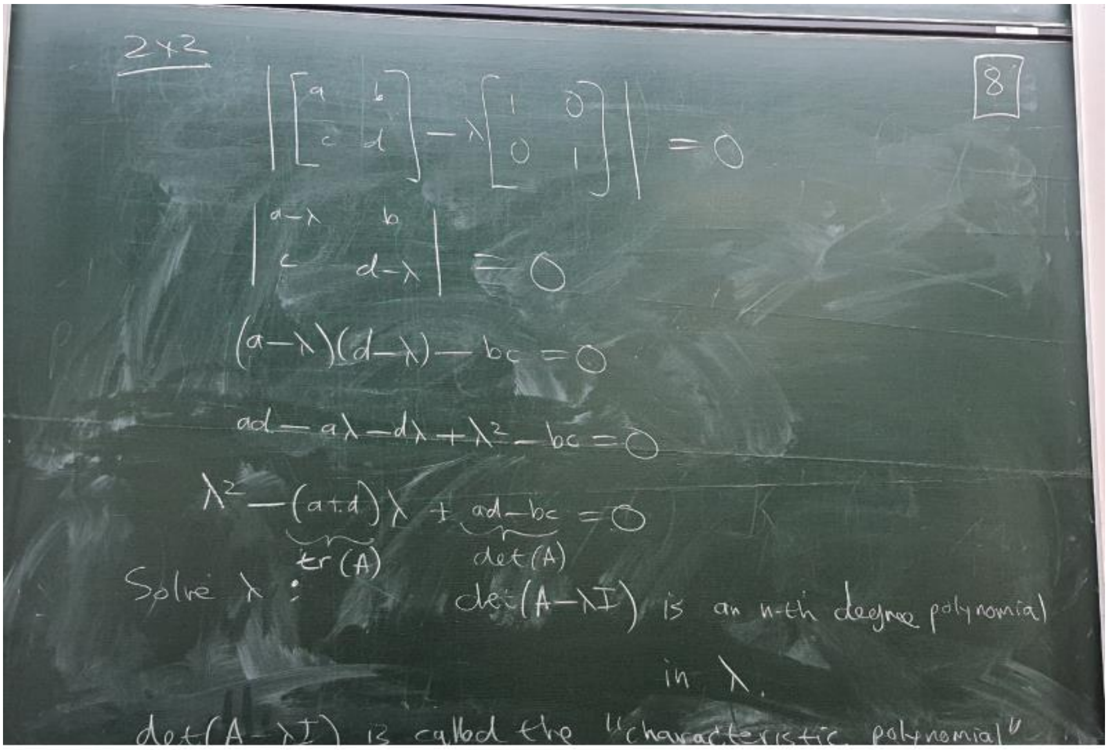
$$A\underline{x} - \lambda\underline{x} = \underline{0}$$

$$(A - \lambda I)\underline{x} = \underline{0}$$

$\underline{x}$ must be a basis vector for null-space of $(A - \lambda I)$

Only a non-null $\underline{x}$, if $A - \lambda I$ is singular.

Want this $\det(A - \lambda I) = 0$



2x2

$\boxed{8}$

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

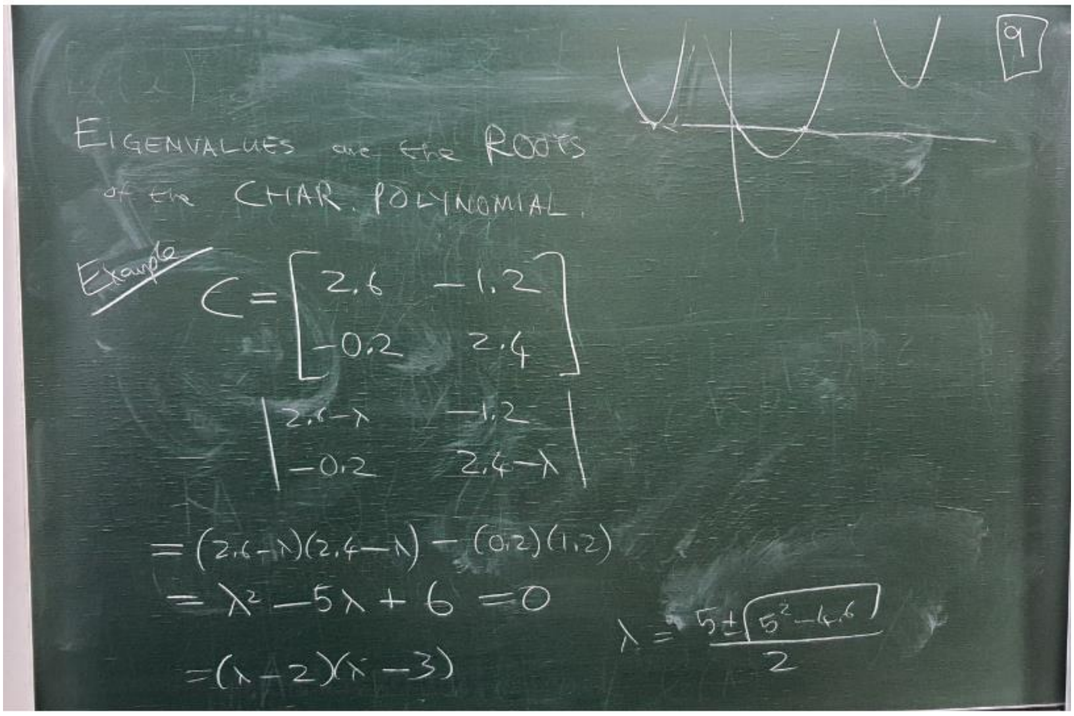
$$(a-\lambda)(d-\lambda) - bc = 0$$

$$ad - a\lambda - d\lambda + \lambda^2 - bc = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{tr(A)}\lambda + \underbrace{ad - bc}_{det(A)} = 0$$

Solve λ : $\det(A - \lambda I)$ is an n-th degree polynomial in λ.

$\det(A - \lambda I)$ is called the "characteristic polynomial"



EIGENVALUES are the ROOTS of the CHAR. POLYNOMIAL.

Example

$$C = \begin{bmatrix} 2.6 & -1.2 \\ -0.2 & 2.4 \end{bmatrix}$$

$$\begin{vmatrix} 2.6-\lambda & -1.2 \\ -0.2 & 2.4-\lambda \end{vmatrix}$$

$$= (2.6-\lambda)(2.4-\lambda) - (0.2)(1.2)$$

$$= \lambda^2 - 5\lambda + 6 = 0$$

$$= (\lambda-2)(\lambda-3)$$

$$\lambda = \frac{5 \pm \sqrt{5^2 - 4.6}}{2}$$