AM214-2023: LECTURE 21

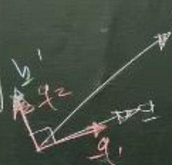
LECTURE 21 — QR AND REDUCED QR-DECOMP.

1

Q is $n \times n$ orthogonal: $Q^TQ = QQ^T = I$

GRAM-SCHMIDT ORTH. $\equiv$ QR DECOMP.

$$A = \begin{bmatrix} | & | & | \\ \underline{a} & \underline{b} & \underline{c} \\ | & | & | \end{bmatrix}$$



$A = QR$

$$Q = \begin{bmatrix} | & \\ \underline{q}_1 & \underline{q}_2 \\ | & \end{bmatrix}$$

$$R = \begin{bmatrix} & & \\ 0 & & \\ 0 & 0 & \end{bmatrix}$$

$$\underline{q}_1 = \frac{\underline{a}}{\|\underline{a}\|}$$

$$\underline{b}' = \underline{b} - \underline{q}_1 \boxed{\underline{q}_1^T \underline{b}}$$

$$\underline{q}_2 = \frac{\underline{b}'}{\|\underline{b}'\|}$$

2

$$\underline{c}' = \underline{c} - \underline{q}_1 \boxed{\underline{q}_1^T \underline{c}} - \underline{q}_2 \boxed{\underline{q}_2^T \underline{c}}$$

$A = QR$

$$R = Q^T A = \begin{bmatrix} — & \underline{q}_1^T & — \\ — & \underline{q}_2^T & — \\ — & \underline{q}_3^T & — \end{bmatrix} \begin{bmatrix} \underline{a} & \underline{b} & \underline{c} \end{bmatrix}$$

$$= \begin{bmatrix} \overset{\|\underline{a}\|}{\cancel{\underline{q}_1^T \underline{a}}} & \boxed{\underline{q}_1^T \underline{b}} & \boxed{\underline{q}_1^T \underline{c}} \\ 0 & \overset{\|\underline{b}'\|}{\cancel{\underline{q}_2^T \underline{b}}} & \boxed{\underline{q}_2^T \underline{c}} \\ 0 & 0 & \underset{\|\underline{c}'\|}{\cancel{\underline{q}_3^T \underline{c}}} \end{bmatrix}$$

$$\underline{q}_1^T \underline{a} = \left(\frac{\underline{a}}{\|\underline{a}\|}\right)^T \underline{a} = \frac{1}{\|\underline{a}\|} \underline{a}^T \underline{a}$$

$$= \frac{\|\underline{a}\|^2}{\|\underline{a}\|} = \|\underline{a}\|$$

$\boxed{3}$

Example: Do QR-decomp

$$A = \begin{array}{c} \begin{bmatrix} 3 & 6 & 2 \\ 4 & 8 & -14 \\ 0 & 10 & 5 \end{bmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \underline{a} & \underline{b} & \underline{c} \end{array} \end{array}, \qquad Q = \begin{bmatrix} 3/5 & 0 & 4/5 \\ 4/5 & 0 & -3/5 \\ 0 & \frac{5}{5} & 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 0 & 4 \\ 4 & 0 & -3 \\ 0 & 5 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 5 & 10 & -10 \\ 0 & 10 & 5 \\ 0 & 0 & 10 \end{bmatrix}$$

Step 1

$$q_1 = \frac{\underline{a}}{\|\underline{a}\|} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

Step 2:

$$\underline{b}' = \underline{b} - q_1 \overset{\frac{50}{5}}{(q_1^T \underline{b})}$$

$$= \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} - \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix} \left(\cancel{\frac{1}{5}\begin{bmatrix} 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix}} \right) = \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} 10$$

$\boxed{4}$

$$= \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} - \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \qquad \|\underline{b}'\| = 10$$

$$q_2 = \frac{1}{10}\begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

Step 3:

$$\underline{c}' = \underline{c} - q_1(q_1^T \underline{c}) - q_2(q_2^T \underline{c})$$

$$= \begin{bmatrix} 2 \\ -14 \\ 5 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \left(\cancel{\frac{1}{5}\begin{bmatrix} 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -14 \\ 5 \end{bmatrix}} \right) (-10)$$

$$- \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left(\cancel{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -14 \\ 5 \end{bmatrix}} \right) (5)$$

$\boxed{5}$

$$= \begin{bmatrix} 2 \\ -14 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ -6 \\ 0 \end{bmatrix} \qquad \|s'\| = 10$$

$$q_3 = \frac{1}{10}\begin{bmatrix} 8 \\ -6 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$$

Solve a system:

$$A\underline{x} = \underline{b}$$

$$QR\underline{x} = \underline{b}$$

$$\boxed{R\underline{x} = Q^T\underline{b}}$$

$\boxed{6}$

REDUCED Q's

$$\bar{Q} = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} \longleftarrow \text{portrait shaped.}$$

PROPERTIES

- $\bar{Q}^T\bar{Q} = \begin{bmatrix} — & q_1^T & — \\ — & q_2^T & — \end{bmatrix}\begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

  $\bar{Q}^T\bar{Q} = I$

- $\bar{Q}\bar{Q}^T \neq I$

7

Proj. mat. on columns of $\bar{Q}$:

$$P = A(A^TA)^{-1}A^T$$
$$= \bar{Q}(\cancel{\bar{Q}^T\bar{Q}})^{-1}\bar{Q}^T$$
$$= \bar{Q}\bar{Q}^T$$

REDUCED QR-DECOMP.

$$A \xleftarrow{\text{portrait}} \quad A = \bar{Q}R$$

Get LS-solution of overdet. system.

$$A\underline{x} = \underline{b}, \text{ but } \underline{b} \notin \mathcal{C}(A)$$

8

Normal equations:

$$\underbrace{A^TA}\tilde{\underline{x}} = A^T\underline{b}$$

~~LU = b'~~

$$A = \bar{Q}R$$
$$(\bar{Q}R)^T\bar{Q}R\tilde{\underline{x}} = (\bar{Q}R)^T\underline{b}$$
$$R^T\underbrace{\bar{Q}^T\bar{Q}}_{I}R\tilde{\underline{x}} = R^T\bar{Q}^T\underline{b}$$

$$R = \begin{bmatrix} \|a'\| & * & * \\ 0 & \|b'\| & * \\ 0 & 0 & \|c'\| \end{bmatrix}$$

R^{-1} exists always.

Premult. by $(R^T)^{-1}$: $\boxed{R\tilde{\underline{x}} = \bar{Q}^T\underline{b}}$

$>> \quad x = A \backslash \underline{b}$

$A\underline{x} = \underline{b}$

$\underline{x} = A^{-1}\underline{b}$

$= A \backslash \underline{b}$

$\cancel{\underline{b}/A}$

Example

Find LS sol of: $A\underline{x} = \underline{b}$

$$A = \begin{bmatrix} -1 & 5 \\ 4 & -11 \\ 8 & -4 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} -14 \\ 1 \\ 9 \end{bmatrix}$$

$$R\underline{x} = Q^T \underline{b} = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$$

$$9\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \text{"}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 10/9 \\ -3/9 \end{bmatrix}$$

$y = -\frac{1}{3}$,