

LECTURE 19 LEAST-SQ-FORMULAE,
ORTHOOGONALITY

1

Solving overdetermined systems

$$A\tilde{x} = \tilde{b}, \text{ but } \tilde{b} \notin \text{col}(A)$$

$$\tilde{x} = (A^T A)^{-1} A^T \tilde{b}, \text{ calculate inverse.}$$

$$(A^T A) \tilde{x} = A^T \tilde{b}$$

$$LU \tilde{x} = \tilde{b}'$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$



FORMULAE FOR FITTING A LINE

2

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

N points; $(x_j, y_j), j=1, 2, \dots, N$

Fit: $y = ax + b$

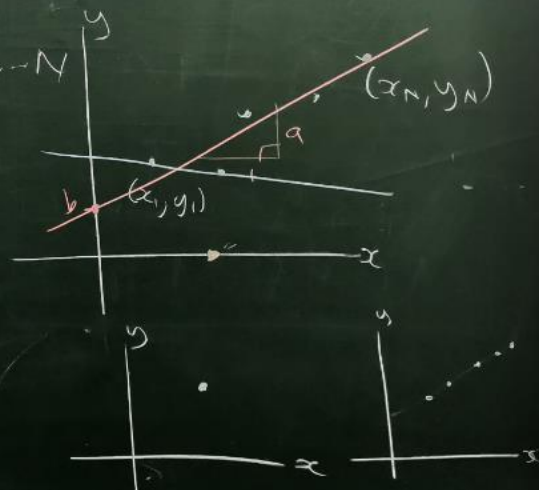
Overdetermined system:

$$ax_1 + b = y_1$$

$$ax_2 + b = y_2$$

$$\vdots$$

$$ax_N + b = y_N$$



$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

3

$$\tilde{x} = (A^T A)^{-1} A^T b$$

$$A^T A = \begin{bmatrix} x_1 & x_2 & \dots \\ 1 & 1 & \dots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n x_j^2 & \sum x \\ \sum x & N \end{bmatrix}$$

shorten $\sum x^2$

$$A^T b = \begin{bmatrix} x_1 & x_2 & \dots \\ 1 & 1 & \dots \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \sum x y \\ \sum y \end{bmatrix}$$

$$\tilde{x} = \frac{1}{N \sum x^2 - (\sum x)^2} \begin{bmatrix} N & -\sum x \\ -\sum x & \sum x^2 \end{bmatrix} \begin{bmatrix} \sum x y \\ \sum y \end{bmatrix}$$

4

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{N \sum x y - \sum x \sum y}{N \sum x^2 - (\sum x)^2} \\ \frac{\sum x^2 \sum y - \sum x \sum x y}{N \sum x^2 - (\sum x)^2} \end{bmatrix}$$

$\sum x \sum y$

$$f(x) = a + b \sin(x) + c e^{-5x} + d x^2 + \frac{f}{x}$$

← Can do linear regression

$$g(x) = a e^{bx} + \sin(cx) \leftarrow \text{No.}$$

5

Fit $f(x)$ to N -points.

$$a + b \sin(x_1) + c e^{-5x_1} + d x_1^2 + \frac{p}{x_1} = y_1$$

$$\begin{bmatrix}
 1 & \sin(x_1) & e^{-5x_1} & x_1^2 & \frac{1}{x_1} \\
 1 & \sin(x_2) & e^{-5x_2} & &
 \end{bmatrix}
 \begin{bmatrix}
 a \\
 b \\
 c \\
 d \\
 p
 \end{bmatrix}
 =
 \begin{bmatrix}
 y_1 \\
 y_2
 \end{bmatrix}$$

Chapter 9

ORTHOGONALITY

6

$\{a, b, c\}$ is an orthogonal set, if.

$\begin{pmatrix} N \\ 2 \end{pmatrix}$

$$a^T b = 0, \quad b^T c = 0, \quad c^T a = 0$$

$\{q_1, q_2, q_3\}$ is an orthonormal set,



$$\text{if } q_1^T q_2 = 0, \quad q_2^T q_3 = 0, \quad q_3^T q_1 = 0$$

$$q_1^T q_1 = 1, \quad q_2^T q_2 = 1, \quad q_3^T q_3 = 1$$

Orthogonal matrix ($n \times n$)

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$$Q = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ - & q_3^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} q_1^T q_1 & q_1^T q_2 & q_1^T q_3 \\ q_2^T q_1 & q_2^T q_2 & q_2^T q_3 \\ q_3^T q_1 & q_3^T q_2 & q_3^T q_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

PROPERTIES

8

① $Q^T Q = I$ $[Q^{-1} = Q^T]$

② $Q^T = Q^{-1}$

③ Premult by Q preserves lengths.

[it is a rotation, or a reflection, or a combination]

④ Premultiplication by Q preserves angles.

⑤ Q is full rank.

Proofs:

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$$\textcircled{3} \quad \underline{b}' = Q\underline{b}$$

$$\|\underline{b}'\|^2 = (Q\underline{b})^T (Q\underline{b})$$

$$= \underline{b}^T (Q^T Q) \underline{b} = \underline{b}^T \underline{b} = \|\underline{b}\|^2$$

$$\textcircled{4} \quad \underline{b}' = Q\underline{b}, \quad \underline{c}' = Q\underline{c}$$

$$\underline{b}^T \underline{c} = bc \cos \theta, \quad (\underline{b}')^T \underline{c}' = b'c' \cos \theta'$$

$$(\underline{b}')^T \underline{c}' = (Q\underline{b})^T (Q\underline{c})$$

$$= \underline{b}^T Q^T Q \underline{c} = \underline{b}^T \underline{c}$$

$$b'c' \cos \theta' = bc \cos \theta \rightarrow \theta = \theta'$$



Four Decomps:

$\underline{L}\underline{U}$, QR , $SALS'$, $U\Sigma V^T$ 10

$$A = \overset{\text{orthogonal}}{Q} \overset{\text{upper triangular}}{R}$$

$(n \times n)$ $(n \times n)$ $(n \times n)$

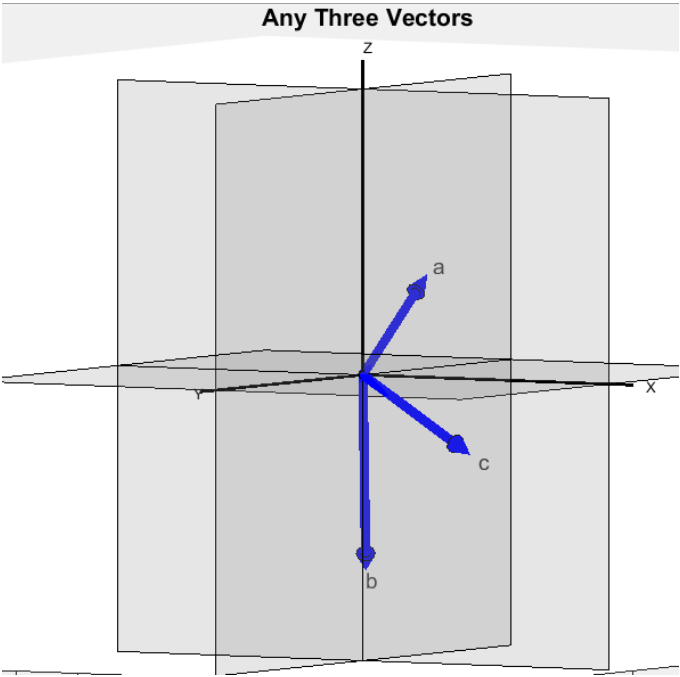
Solve $A\underline{x} = \underline{b}$

$$A = QR$$

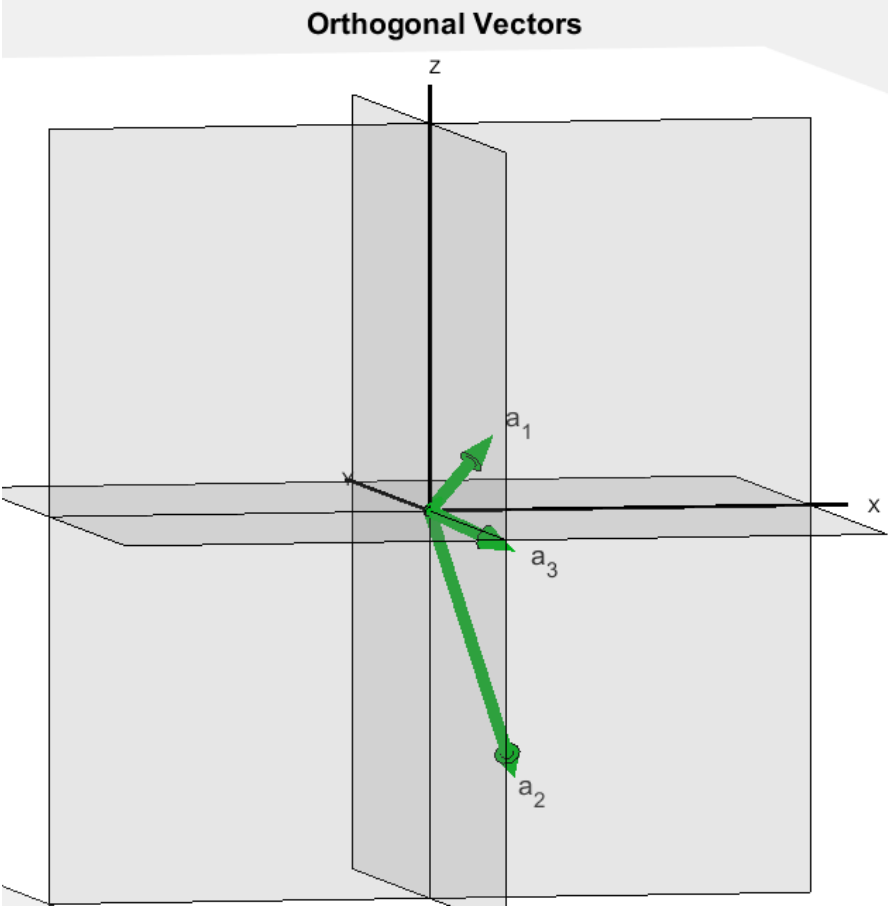
$$QR\underline{x} = \underline{b}$$

$$R\underline{x} = Q^T \underline{b}$$

Any three independent vectors:



Three ORTHOGONAL vectors (mutually orthogonal, but any lengths):



Three ORTHONORMAL vectors (mutually orthogonal unit vectors):

