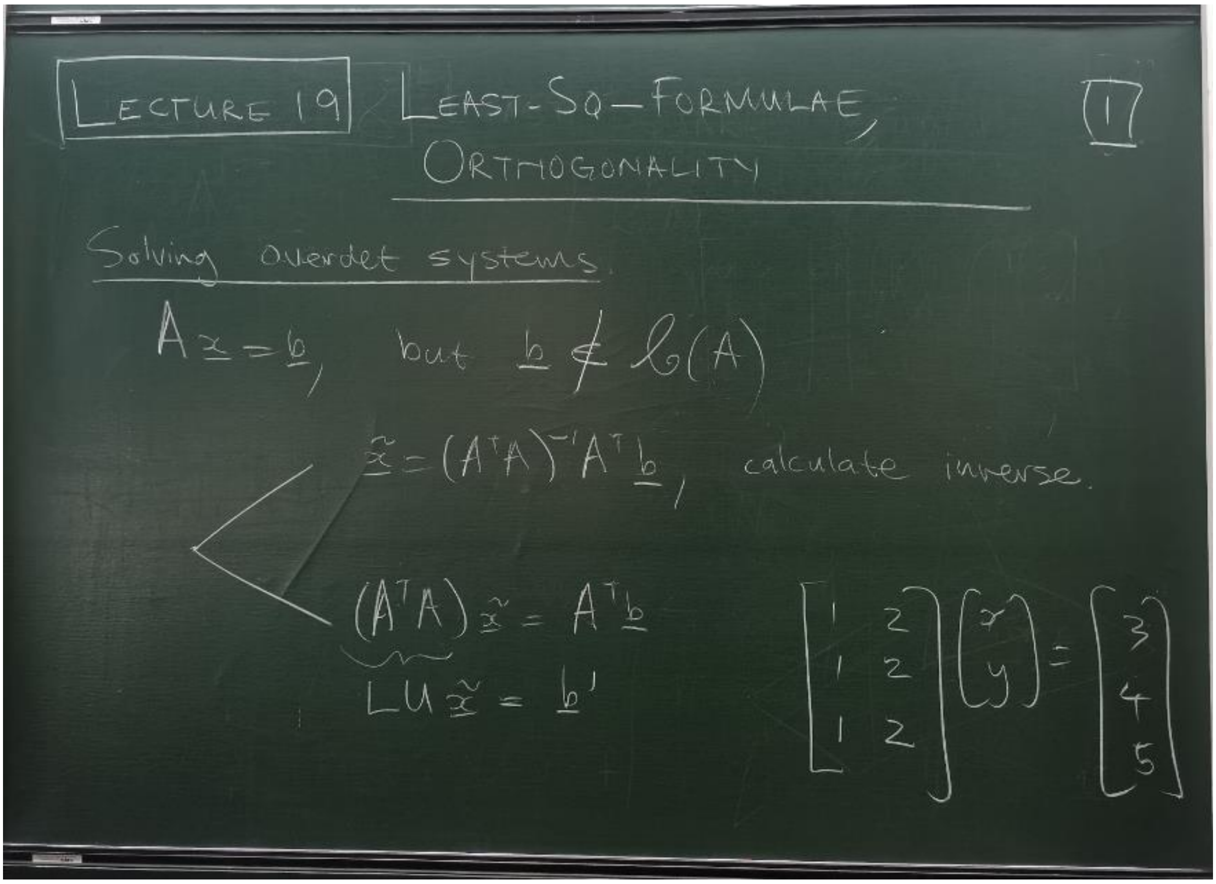
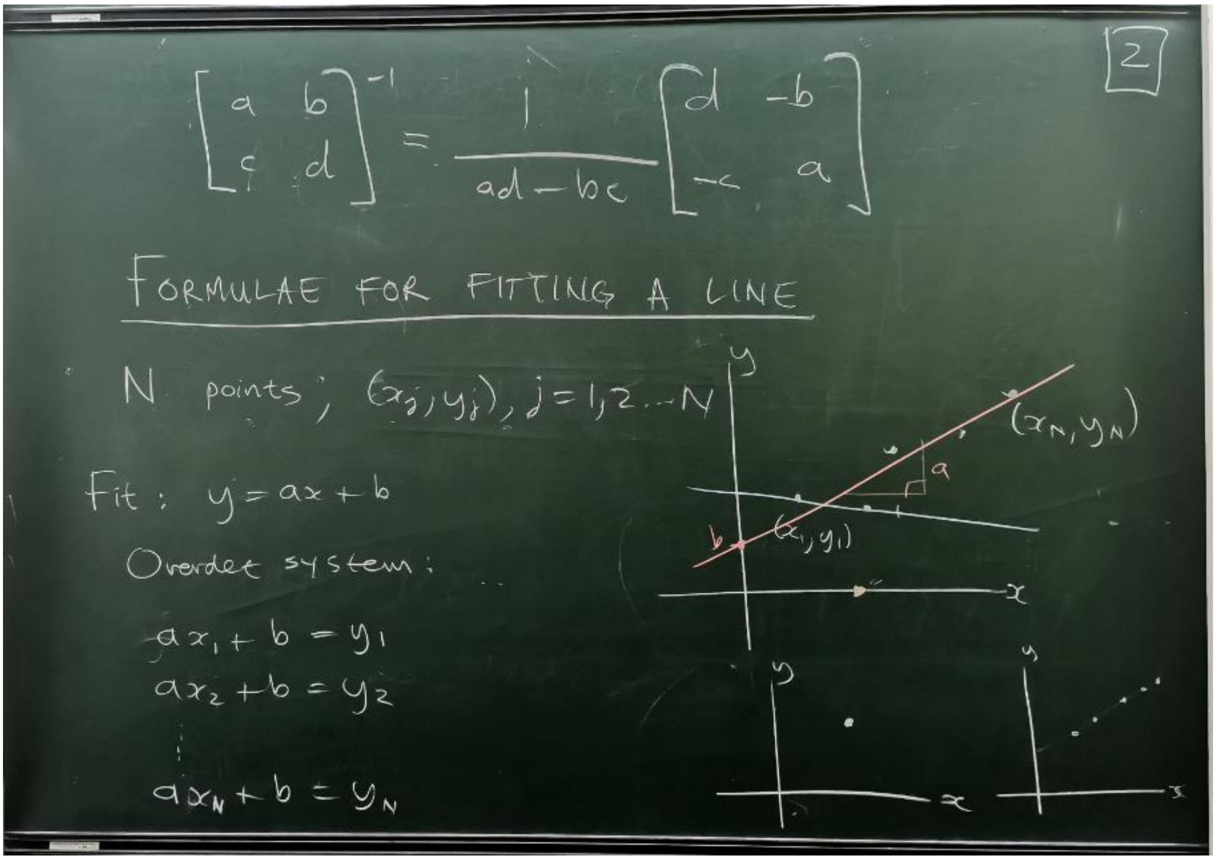
# AM214-2023: LECTURE 19



$\boxed{3}$

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

$$\tilde{\underline{x}} = (A^TA)^{-1}A^T\underline{b}$$

shorten $\sum x^2$

$$A^TA = \begin{bmatrix} x_1 & x_2 & \cdots \\ 1 & 1 & \cdots \end{bmatrix}\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{N} x_j^2 & \sum x \\ \sum x & N \end{bmatrix}$$

$$A^T\underline{b} = \begin{bmatrix} x_1 & x_2 & \cdots \\ 1 & 1 & \cdots \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \sum xy \\ \sum y \end{bmatrix}$$

$\boxed{4}$

$$\tilde{\underline{x}} = \frac{1}{N\sum x^2 - \left(\sum x\right)^2}\begin{bmatrix} N & -\sum x \\ -\sum x & \sum x^2 \end{bmatrix}\begin{bmatrix} \sum xy \\ \sum y \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \dfrac{N\sum xy - \sum x \sum y}{N\sum x^2 - \left(\sum x\right)^2} \\[2ex] \dfrac{\sum x^2 \sum y - \sum x \sum xy}{N\sum x^2 - \left(\sum x\right)^2} \end{bmatrix}$$

$\sum x \quad \sum y$

$$f(x) = a + b\sin(x) + ce^{-5x} + dx^2 + \frac{f}{x}$$

$\leftarrow$ Can do linear regression

$$g(x) = ae^{bx} + \sin(cx) \leftarrow \text{No.}$$

$\boxed{5}$

Fit $f(x)$ to N-points.

$$a + b\sin(x_1) + c\,e^{-5x_1} + d x_1^2 + \frac{f}{x_1} = y_1$$

$$\begin{bmatrix} 1 & \sin(x_1) & e^{-5x_1} & x_1^2 & \frac{1}{x_1} \\ 1 & \sin(x_2) & e^{-5x_2} & & \\ & & & & \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ f \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \\ \end{bmatrix}$$

$\boxed{6}$

# $\boxed{\text{Chapter 9}}$ ORTHOGONALITY

$\{\underline{a}, \underline{b}, \underline{c}\}$ is an orthogonal set, if

$$\underline{a}^T\underline{b} = 0, \quad \underline{b}^T\underline{c} = 0, \quad \underline{c}^T\underline{a} = 0$$

$\binom{N}{2}$

$\{q_1, q_2, q_3\}$ is an orthonormal set,


if $\quad q_1^T q_2 = 0, \quad q_2^T q_3 = 0, \quad q_3^T q_1 = 0$

$\qquad q_1^T q_1 = 1, \quad q_2^T q_2 = 1 \quad q_3^T q_3 = 1$

$\boxed{7}$

## Ortogonal matrix: $(n \times n)$

$$Q = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}$$

$$Q^TQ = \begin{bmatrix} - & q_1^T & - \\ - & q_2^T & - \\ - & q_3^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} q_1^Tq_1 & q_1^Tq_2 & q_1^Tq_3 \\ q_2^Tq_1 & q_2^Tq_2 & \\ & & \ddots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{8}$

## PROPERTIES

- ① $Q^TQ = I$ $\quad [\, Q^{-1} = Q^T \,]$
- ② $Q^T = Q^{-1}$
- ③ Premult by $Q$ preserves lengths.

  [ It is a rotation, or a reflection, or a combination ]
- ④ Premultiplication by $Q$ preserves angles.
- ⑤ $Q$ is full rank.

$\boxed{9}$

Proofs:

$\odot_3$ $\quad \underline{b}' = Q\underline{b}$

$$\|\underline{b}'\|^2 = (Q\underline{b})^T(Q\underline{b})$$
$$= \underline{b}^T(Q^TQ)\underline{b} = \underline{b}^T\underline{b} = \|\underline{b}\|^2$$

$\odot_4$ $\quad \underline{b}' = Q\underline{b}, \quad \underline{c}' = Q\underline{c}$

$$\underline{b}^T\underline{c} = bc\cos\theta, \quad (\underline{b}')^T\underline{c}' = b'c'\cos\theta'$$


$$(\underline{b}')^T\underline{c}' = (Q\underline{b})^T(Q\underline{c})$$
$$= \underline{b}^T\cancel{Q^TQ}\underline{c} = \underline{b}^T\underline{c}$$
$$\cancel{b'}\cancel{c'}\cos\theta' = \cancel{b}\cancel{c}\cos\theta \longrightarrow \theta = \theta'$$

$\boxed{10}$

FOUR DECOMP: $LU$, $QR$, $S\Lambda S^{-1}$, $U\Sigma V^T$

$$A = QR$$

orthogonal ($Q$); upper triangular ($R$).

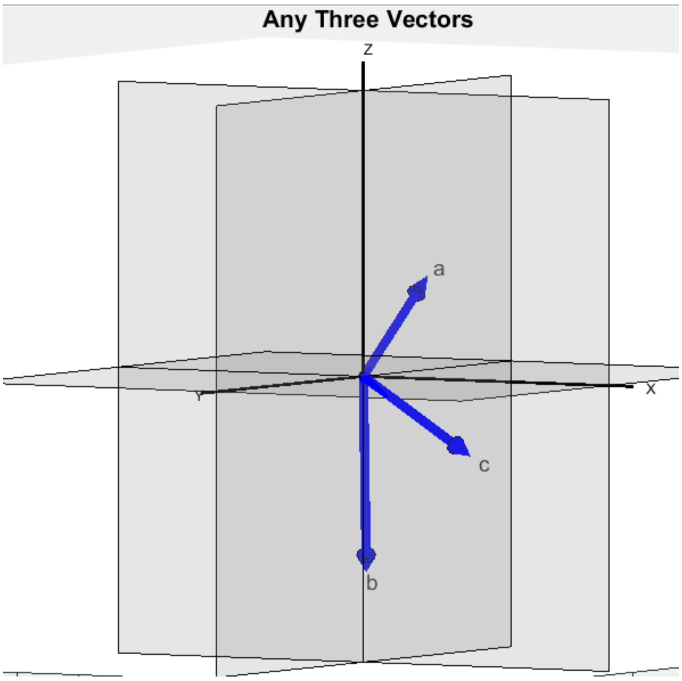
$A$: $(n\times n)$, $Q$: $(n\times n)$, $R$: $(n\times n)$

Solve $A\underline{x} = \underline{b}$

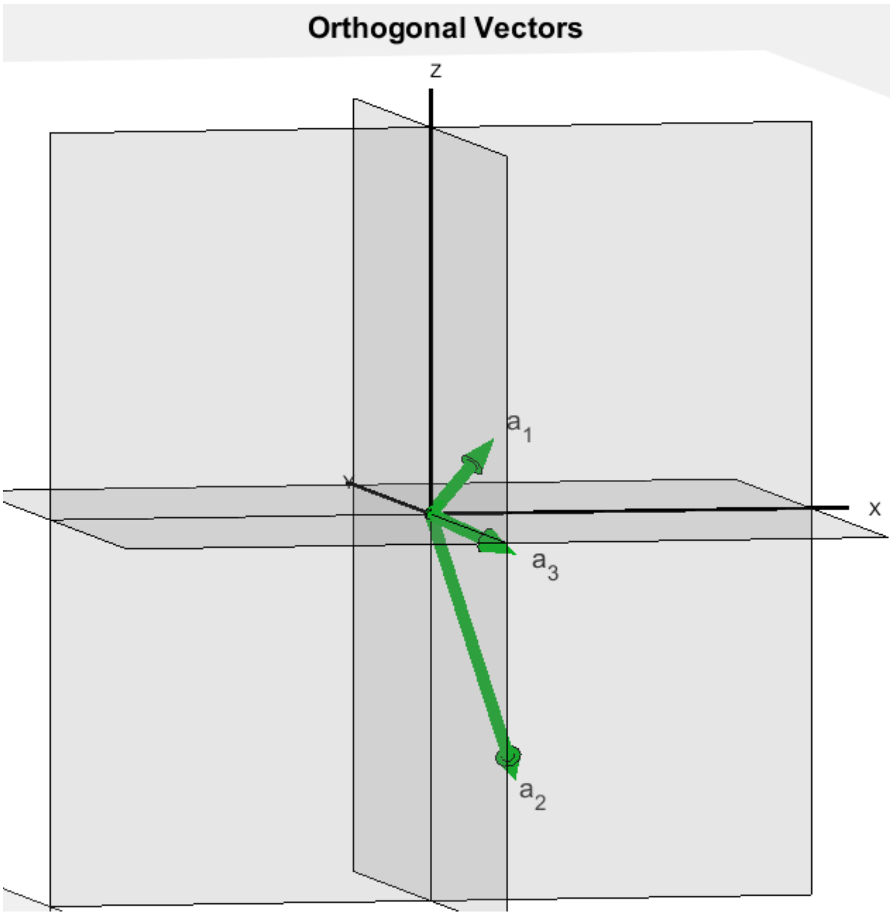
$$QR\underline{x} = \underline{b}$$
$$R\underline{x} = Q^T\underline{b}$$

$A = QR$

Any three independent vectors:




Three ORTHOGONAL vectors (mutually orthogonal, but any lengths):



Three ORTHONORMAL vectors (mutually othogonal unit vectors):