AM214-2023: LECTURE 18

LECTURE 18 LEAST SQUARES SOLUTIONS

1

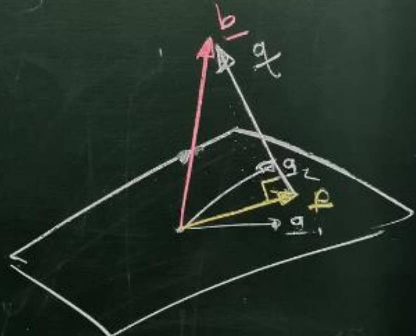
$A\underline{x} = \underline{b}$, but $\underline{b} \notin \mathcal{C}(A)$

$$0.5x + 2y = 1.5$$
$$2x + y = 1$$
$$2x - y = 2$$

$$\begin{bmatrix} 0.5 & 2 \\ 2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1 \\ 2 \end{bmatrix}$$

Rather solve $A\tilde{\underline{x}} = \underline{p}$

$\underline{p}$ as close as possible to $\underline{b}$



(1) $\underline{p} = A\tilde{\underline{x}}$

(2) $\underline{b} = \underline{p} + \underline{q}$, $\underline{q} = \underline{b} - \underline{p}$

2

(3) $\underline{q}$ orthogonal to $\mathcal{C}(A)$

$$A^T \underline{q} = \underline{0}$$

$$A^T(\underline{b} - A\tilde{\underline{x}}) = \underline{0}$$

$$\boxed{A^T A \tilde{\underline{x}} = A^T \underline{b}}$$

$$\tilde{\underline{x}} = (A^T A)^{-1} A^T \underline{b}$$

Least squares solution.

If you want $\underline{p}$:

$$\underline{p} = A\tilde{\underline{x}} = A(A^T A)^{-1} A^T \underline{b}$$

$\|\underline{b} - \underline{p}\|^2$ is minimum

$\boxed{3}$

$(b_1-p_1)^2+(b_2-p_2)^2 \dots$ is minimum.

The "normal equations":

$$A\underline{x}=\underline{b}, \quad \text{but } \underline{b}\notin \mathcal{C}(A)$$

$$\underbrace{A^TA}\,\tilde{\underline{x}}=A^T\underline{b}$$

$$LU=A^TA$$

$\boxed{4}$

FITTING OF CURVES:

Points (x_j, y_j), $j=1,2,3\dots N$

Fit a line:

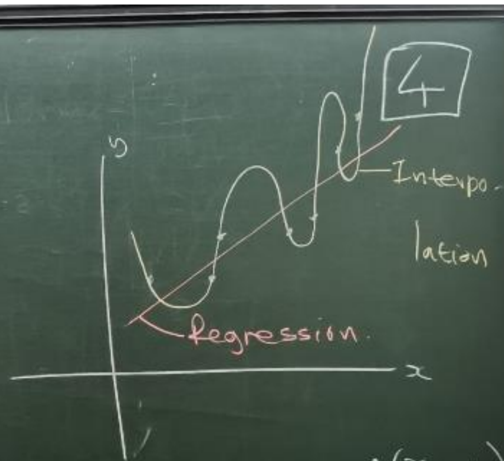
$$y=ax+b$$

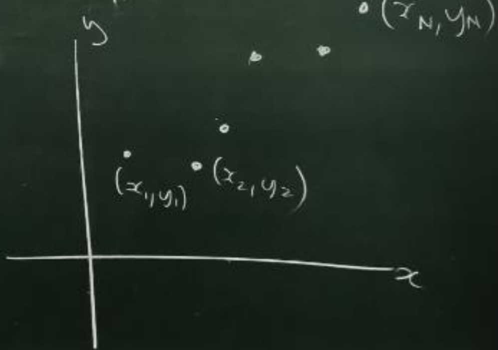
$$y_1=ax_1+b \qquad \tilde{\underline{x}}=\begin{bmatrix}a\\b\end{bmatrix}$$

$$y_2=ax_2+b$$

$$\vdots$$

$$y_N=ax_N+b$$





5

$$\underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\underline{c}} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}}_{\underline{y}}$$

$$A\underline{c} = \underline{y}\,, \quad y \notin \mathcal{C}(A)$$

Normal equ's.

$$A^T A \underbrace{\underline{c}}_{\tilde{\underline{c}}} = A^T \underbrace{\underline{y}}_{\underline{b}}$$

Fit a parab.

$$y = ax^2 + bx + c$$

$$y_1 = ax_1^2 + bx_1 + c$$

$$y_2 = ax_2^2 + bx_2 + c$$

$$\vdots$$

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ & & \\ & & \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \\ \end{bmatrix}$$

6

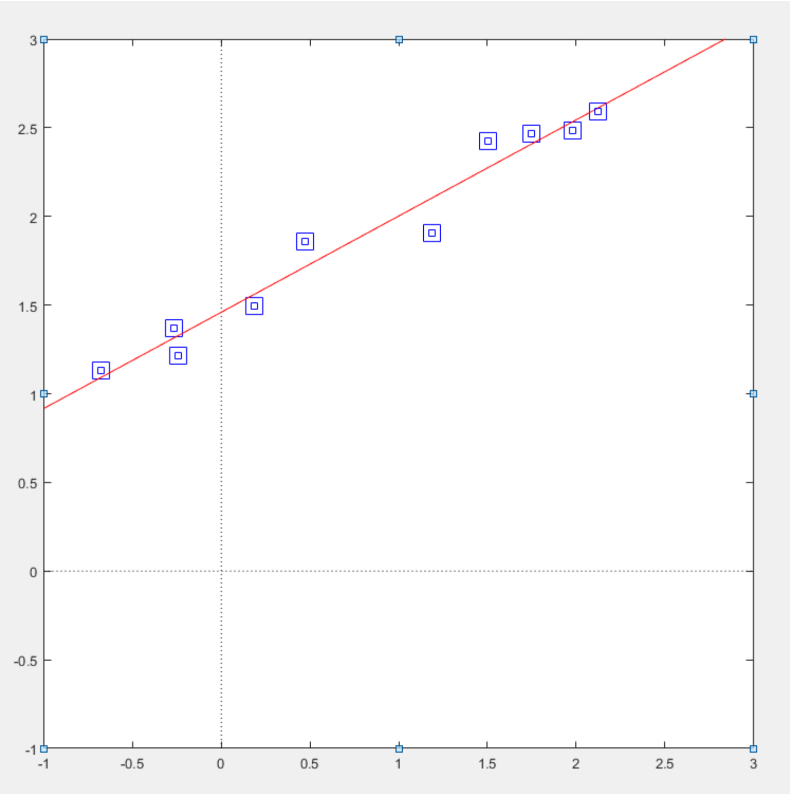
Fit a plane.

$$ax + by + cz = d$$

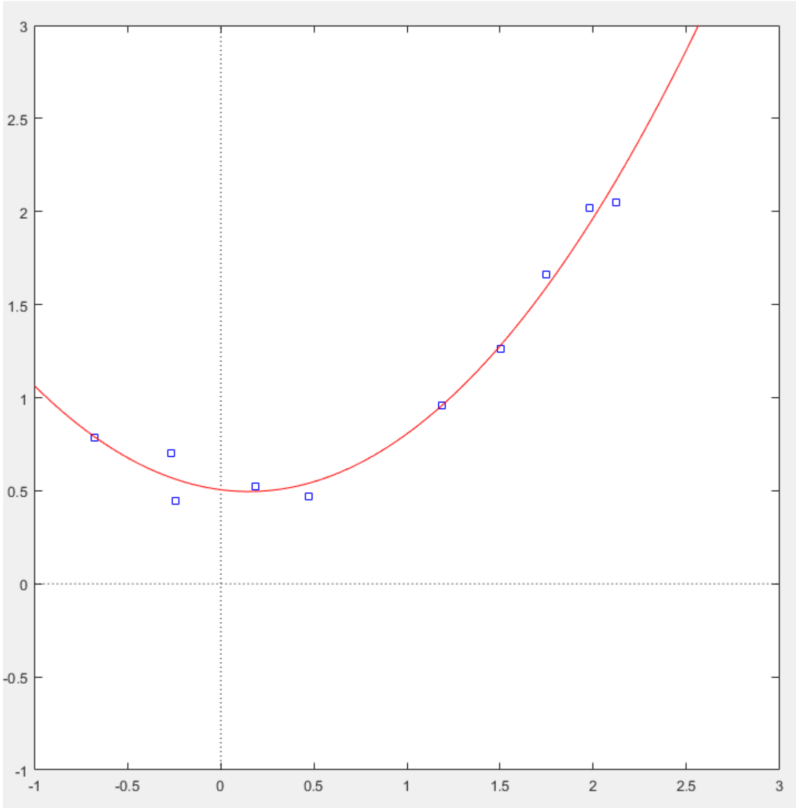
$$z = \cancel{-\frac{a}{c}}x \cancel{-\frac{b}{c}}y + \frac{d}{c}$$

$$z = \bar{a}x + \bar{b}y + \bar{c}$$

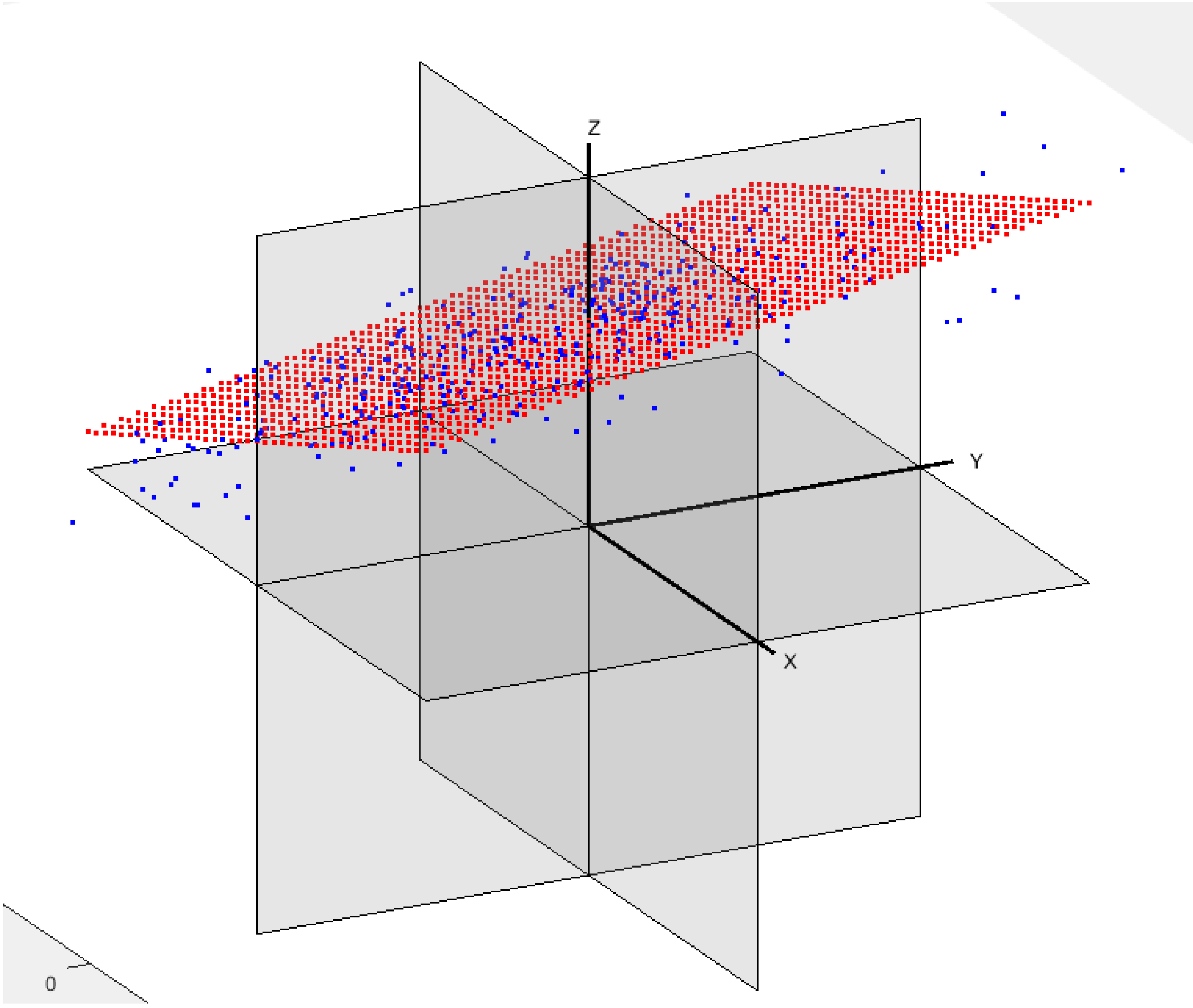
$$\bar{a}x_1 + \bar{b}y_1 + \bar{c} = z_1$$



3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
1.5
2
2.5
3



3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
1.5
2
2.5
3


Z
Y
X
0