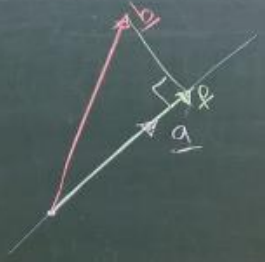
AM214-2023: LECTURE 15

LECTURE 15 — PROJECTIONS AND REFLECTIONS

1

$$P = \frac{\underline{a}\,\underline{a}^T}{\underline{a}^T\underline{a}} = \frac{1}{\underline{a}^T\underline{a}}\left(\underline{a}\,\underline{a}^T\right)$$

$$\boxed{\underline{p} = P\underline{b}} = \underline{a}\left(\frac{1}{\underline{a}^T\underline{a}}\right)\underline{a}^T = \underline{a}\left(\underline{a}^T\underline{a}\right)^{-1}\underline{a}^T$$



$$(\lambda\,\underline{a})\underline{b} = (\underline{a}\,\lambda)\underline{b}$$

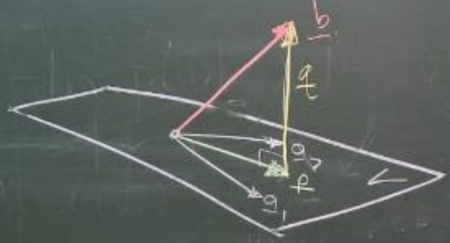
$$\underline{a}\,\underline{a}^T = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} a_1a_1 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2a_2 & a_2a_3 \\ & & \end{bmatrix}$$

PROJECTION ON A PLANE IN $\mathbb{R}^3$

2

$\text{basis}(V) = \{\underline{a}_1, \underline{a}_2\}$

Pack basisvectors in a matrix



$$A = \begin{bmatrix} | & | \\ \underline{a}_1 & \underline{a}_2 \\ | & | \end{bmatrix}$$

① $$\underline{p} = \lambda\underline{a}_1 + \mu\underline{a}_2 = \underbrace{\begin{bmatrix} | & | \\ \underline{a}_1 & \underline{a}_2 \\ | & | \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} \lambda \\ \mu \end{bmatrix}}_{\underline{x}}$$

$$\underline{p} = A\overset{?}{\underline{x}}$$

② $\underline{p} + \underline{q} = \underline{b}$, $\underline{q} = \underline{b} - \underline{p}$

3

③ $\underline{a}_1^T \underline{q} = 0$

$\underline{a}_2^T \underline{q} = 0$

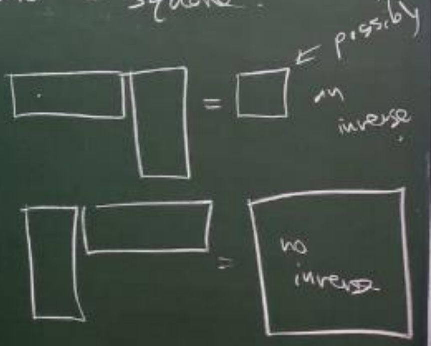
$$\begin{bmatrix} \text{—} \underline{a}_1^T \text{—} \\ \text{—} \underline{a}_2^T \text{—} \end{bmatrix} \underline{q} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$A^T \underline{q} = \underline{0}$

$A^T(\underline{b} - A\underline{x}) = \underline{0}$

$B^T B$ = square

$B B^T$ = square.



Solve $\underline{x}$:

$A^T \underline{b} - A^T A \underline{x} = \underline{0}$

$(A^T A)\underline{x} = A^T \underline{b}$

Premult. by $(A^T A)^{-1}$:

$\therefore \underline{x} = (A^T A)^{-1} A^T \underline{b}$

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$$\underline{p} = A\underline{x} = \underbrace{A(A^T A)^{-1} A^T}_{P} \underline{b}$$

$$\boxed{P = A(A^T A)^{-1} A^T}$$

Properties of P:

① P does not depend on the lengths of $\underline{a}_1$ or $\underline{a}_2$

② $P^T = P$

$$P^T = \left(A(A^T A)^{-1} A^T\right)^T$$
$$= (A^T)^T \left[(A^T A)^{-1}\right]^T A^T$$

$\boxed{5}$

$$= A\left[(A^TA)^T\right]^{-1}A^T$$

$$= A(A^TA^{TT})^{-1}A^T = A(A^TA)^{-1}A^T$$

③ P is idempotent

$$P^2 = P$$

$$P^2 = \left(A(A^TA)^{-1}A^T\right)\left(A(A^TA)^{-1}A^T\right)$$

$$= A(A^TA)^{-1}\underbrace{(A^TA)(A^TA)^{-1}}A^T$$

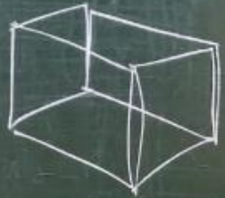
$$= A(A^TA)^{-1}\, I\, A^T$$

④ $P_1 = I - P$ is also a projection matrix.

$$(P_1)^2 = (I-P)(I-P) = I - P = P_1$$

$\boxed{6}$

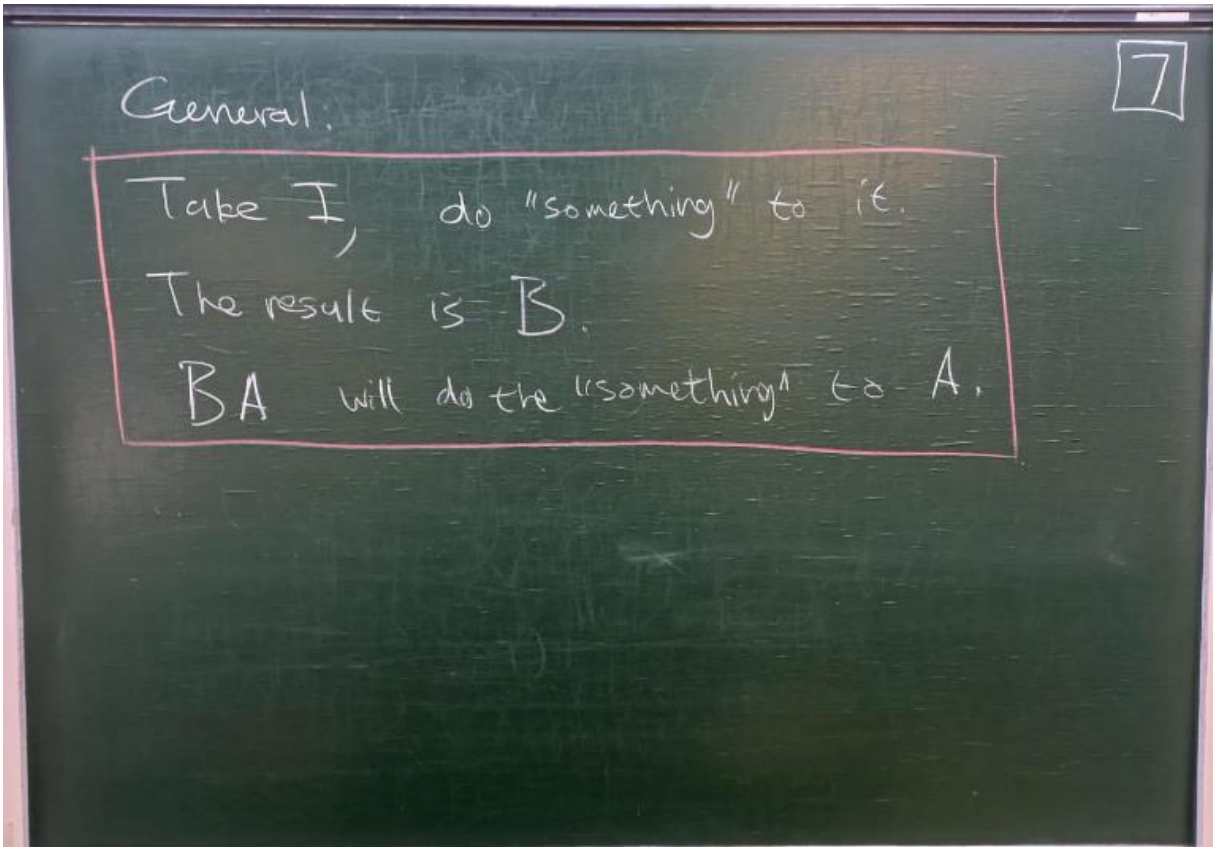
$$P\begin{bmatrix} | & | & | & | \\ \underline{b}_1 & \underline{b}_2 & \underline{b}_3 & \underline{b}_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} P\underline{b}_1 & P\underline{b}_2 & \dots & P\underline{b}_n \end{bmatrix}$$

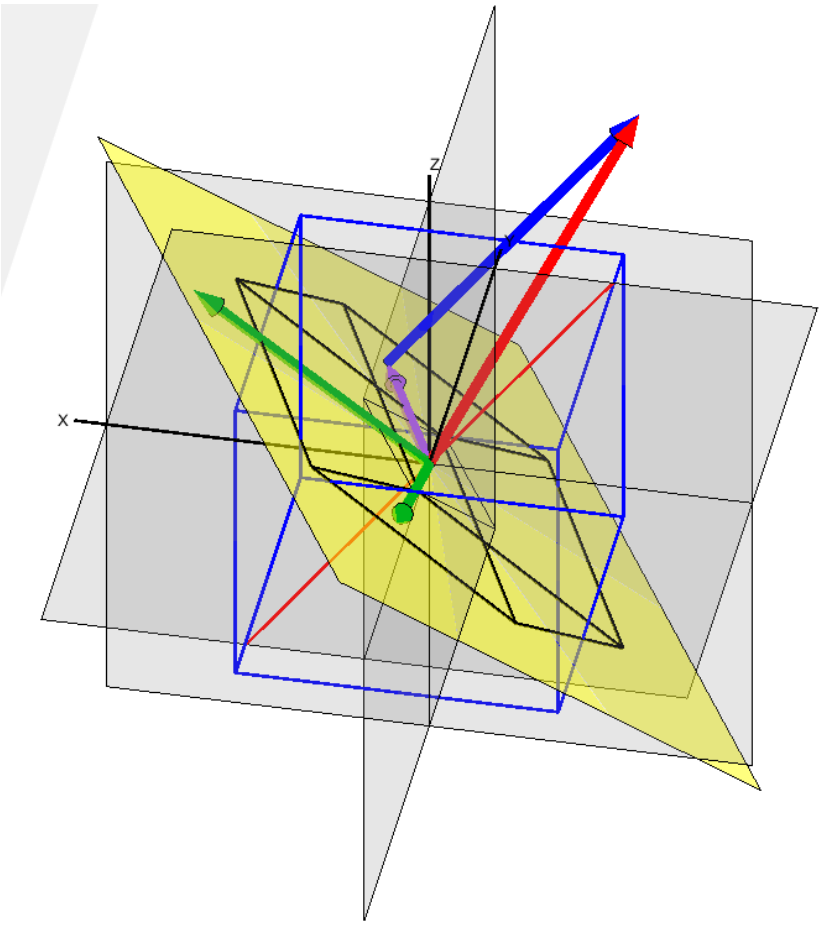


Columns of P

$$P\underline{i} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = 1p_1 + 0p_2 + 0p_3$$
$$= p_1$$

$$P[\underline{i}\ \underline{j}\ \underline{k}] = PI = P.$$


7
General:
Take I, do "something" to it.
The result is B.
BA will do the "something" to A.


z
x