

AM214-2023: LECTURE 14

LECTURE 14 PROJECTIONS 1

A is $m \times n$

$\mathcal{R}(A) = \{ b \in \mathbb{R}^m \mid b = zA, z \in \mathbb{R}^n \}$

$\mathcal{N}(A) = \{ z \in \mathbb{R}^n \mid Az = 0 \}$

$b \in \mathcal{R}(A)$
 $z \in \mathcal{N}(A)$
 $b^T z = 0$

$\mathcal{R}(A) \perp \mathcal{N}(A)$

Row space and Null space form orthogonal complement spaces in \mathbb{R}^n

$\mathcal{R}(A)$ and $\mathcal{L}(A)$ form orth. compl. spaces in \mathbb{R}^m . 2

$\mathcal{L}(A)$ $\left[\begin{array}{cccccccc} | & | & | & | & | & | & | & | \end{array} \right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

PROJECTIONS IN \mathbb{R}^3

Proj. on a line spanned by \underline{a}

About P

- ① $f = \lambda a$
- ② $f + q = b$
- ③ $a^T q = 0$

Using LA

- ②: $q = b - f$
- ①: $q = b - \lambda a$
- ③: $a^T q = a^T (b - \lambda a) = 0$

$$a^T b - \lambda a^T a = 0$$



Using trig

$$\frac{p}{b} = \cos \theta$$

$$p = b \cos \theta$$

$$f = p \frac{a}{a} = a \frac{b \cos \theta}{a}$$

$$= \frac{b \cos \theta}{a} a$$

$$\lambda (a^T a) = a^T b$$

$$\lambda = \frac{a^T b}{a^T a}$$

$$f = a \lambda = a \left(\frac{a^T b}{a^T a} \right) = \frac{1}{a^T a} ((a a^T) b)$$

$$= \frac{1}{a^T a} \left(\begin{array}{|c|} \hline a \\ \hline \end{array} \begin{array}{|c|} \hline a^T \\ \hline \end{array} \begin{array}{|c|} \hline b \\ \hline \end{array} \right)$$

square matrix

$$f = \underbrace{\left(\frac{a a^T}{a^T a} \right)}_P b$$

P is called the projection matrix.

Properties of P

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① $P^2 = P$

$$P^2 = \frac{aa^T}{a^T a} \frac{aa^T}{a^T a} = \frac{1}{(a^T a)^2} [a(a^T a)a^T]$$
$$= \frac{aa^T}{(a^T a)^2} a^T a = \frac{aa^T}{a^T a}$$

P is idempotent

② P does not depend on $\|a\|$.

μa

$$P = \frac{(\mu a)(\mu a^T)}{(\mu a)^T(\mu a)} = \frac{aa^T}{a^T a}$$

$$f = P b$$

$$f = P(Pb) = P^2 b$$

③ $P_1 = I - P$ is also a projection matrix

③ $P_1^2 = (I - P)^2 = (I - P)(I - P)$

$$\begin{aligned} &= I - 2P + P^2 \\ &= I - 2P + P = I - P \end{aligned}$$

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$$q = b - f = b - Pb = (I - P)b$$

P projects on line L

$I - P$ " " plane orthogonal to line L.

$$(P_1 + P)b = Pb + Pb = q + f = b$$
$$I b$$

