AM214-2023: LECTURE 13

$\boxed{\text{Lecture 13}}$ Col + Null Spaces, Applications

$\boxed{1}$

Chemical balancing:

$$2H_2 + O_2 \to 2H_2O$$

$$a\,KMnO_4 + b\,HCl \to c\,KCl + d\,MnCl_2 + e\,H_2O + f\,Cl_2$$

K: $a = c$ $\quad a = c = d$

Mn: $a = d$

O: $4a = e$

H: $b = 2e$

Cl: $b = \overset{a}{\cancel{c}} + 2\overset{a}{\cancel{d}} + 2f = 3a + 2f$

a, b, e, f

$\boxed{2}$

$$A = \begin{array}{cccc} -3 & & 3/4 & \\ \end{array}$$

$$A = \begin{bmatrix} \boxed{4} & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ -3 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} 4 & 0 & -1 & 0 \\ 0 & \boxed{1} & -2 & 0 \\ 0 & 1 & -\frac{3}{4} & -2 \end{matrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{3}{4} & 1 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} \boxed{4} & 0 & -1 & 0 \\ 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & \boxed{\frac{5}{4}} & -2 \end{bmatrix}$$

$$A\underline{x} = \underline{0}$$

$$\begin{bmatrix} \boxed{4} & 0 & -1 & 0 \\ 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & \boxed{\frac{5}{4}} & \underset{\uparrow \atop \lambda}{-2} \end{bmatrix} \begin{bmatrix} a \\ b \\ e \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{5}{4}e - 2\lambda = 0$$

$$e = \frac{8}{5}\lambda$$

$$b - 2\left(\frac{8}{5}\lambda\right) = 0$$

$$b = \frac{16}{5}\lambda$$

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$$4a - \left(\frac{8}{5}\lambda\right) = 0, \quad a = \frac{8}{5}\cdot\frac{1}{4}\lambda = \frac{2}{5}\lambda$$

$$\begin{bmatrix} a \\ b \\ e \\ f \end{bmatrix} = \begin{bmatrix} \frac{2}{5}\lambda \\ \frac{16}{5}\lambda \\ \frac{8}{5}\lambda \\ \lambda \end{bmatrix} \qquad \text{Let } \lambda = 5$$

$a = 2 = c = d$

$b = 16$

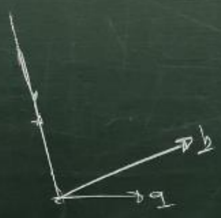
$e = 8$

$f = 5$

$$2\,KMnO_4 + 16\,HCl \longrightarrow$$
$$2\,KCl + 2\,MnCl_2 + 8\,H_2O + 5\,Cl_2$$

$\boxed{4}$

$$6\begin{bmatrix} & 6. & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}$$



$\underline{a}, \underline{b}$

$\underline{a}^T \underline{c} = 0$

$\underline{b}^T \underline{c} = 0$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solve this, c_1, c_2, c_3 must not be fractions.

$\boxed{5}$

$$L = \begin{bmatrix} 1 & 0 \\ \frac{b_1}{a_1} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 - \frac{a_2 b_1}{a_1} & \cdots \end{bmatrix}$$

LU-DECOMPOSITION OF PORTRAIT-MATRIX

$$A = \begin{bmatrix} \boxed{2} & 1 \\ 2 & 5 \\ 4 & 2 \\ 2 & -3 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 3 \\ 7 \\ 6 \\ -1 \end{bmatrix}$$

$$\begin{matrix} 2 & 1 \\ 0 & 4 \\ 0 & 0 \\ 0 & -4 \end{matrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & -1 & \cancel{\alpha}\ 0 & 1 \end{bmatrix}$$

$\boxed{6}$

$$\begin{matrix} 2 & 1 \\ 0 & \boxed{4} \\ 0 & 0 \\ 0 & -4 \end{matrix}$$

$$U = \begin{bmatrix} \boxed{2} & 1 \\ 0 & \boxed{4} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Col space

$$\text{basis}(\mathcal{C}(A)) = \left\{ \begin{bmatrix} 2 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 2 \\ -3 \end{bmatrix} \right\}$$

Null space

$$U\underline{x} = \underline{0} \qquad \begin{bmatrix} 2 & 1 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathcal{N}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

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$Ax=b$

$L\underline{c}=\underline{b}$

$$\begin{bmatrix}1&0&0&0\\1&1&0&0\\2&0&1&0\\1&-1&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}=\begin{bmatrix}3\\7\\6\\-1\end{bmatrix}$$

$c_1=3$

$3+c_2=7, \quad c_2=4$

$2(3)+c_3=6, \quad c_3=0$

$1(3)-1(4)-c_4=-1, \quad c_4=0$

$\underline{c}=\begin{bmatrix}3\\4\\0\\0\end{bmatrix}$ ← There is a solution

$U\underline{x}=\underline{c}$

$$\begin{bmatrix}2&1\\0&4\\0&0\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}3\\4\\0\\0\end{bmatrix}$$

$4x_2=4, \quad x_2=1$

$2x_1+1=3 \quad x_1=1$

$\underline{x}=\begin{bmatrix}1\\1\end{bmatrix}$

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Four Fundamental Spaces of A

$A\in\mathbb{R}^{4\times 7}$

$$\underset{4\times 7}{A}=\underset{4\times 4}{L}\,\underset{4\times 7}{U}$$

$$U=\begin{array}{c}\begin{bmatrix}\boxed{\times}&\times&\times&\times&\times&\times&\times\\0&0&\boxed{\times}&\times&\times&\times&\times\\0&0&0&0&\boxed{\times}&\times&\times\\0&0&0&0&0&0&0\end{bmatrix}\\ \begin{array}{ccccccc}\ \ &\lambda&\ \ &\mu&\ \ &\kappa&\theta\end{array}\end{array}$$

rank(U) — number of independent col's in U.
— " " " rows in U.

$\dim(\mathcal{C}(U))$ = number of pivots $=3$ rank $=3$

$\dim(\mathcal{R}(U))$ = " " $=3$

$\dim(\mathcal{N}(U))=4=7-3$

Left Null space: $\boxed{9}$

$$\mathcal{L}(A) = \{ \underline{c} \in \mathbb{R}^4 \mid \underline{c}^T A = \underline{0}^T, \underline{0}^T \in \mathbb{R}^7 \}$$

$$\underline{c}^T A = \underline{0}^T \xrightarrow{\text{Transpose}} A^T \underline{c} = \underline{0}$$

SUMMARY $\boxed{10}$

A is $m \times n$ with $\text{rank}(A) = r$

Space	Symbol	In which host sp.	Dimension	
Column sp.	$\mathcal{C}(A)$	$\mathbb{R}^m$	r	orthogonal
Left null sp.	$\mathcal{L}(A)$	$\mathbb{R}^m$	$m-r$	
Row sp.	$\mathcal{R}(A)$	$\mathbb{R}^n$	r	
Null sp.	$\mathcal{N}(A)$	$\mathbb{R}^n$	$n-r$	