AM214-2023: LECTURE 12

$\boxed{\text{LECTURE 12}}$ SOLVING RECTANGULAR SYSTEMS
COLUMN + NULL SPACES

$\boxed{1}$

Definitions: $A \in \mathbb{R}^{m \times n}$

Col-space

$$\mathcal{C}(A) = \{ \underline{b} \in \mathbb{R}^m \mid \underline{b} = A\underline{x},\ \underline{x} \in \mathbb{R}^n \}$$

Null space

$$\mathcal{N}(A) = \{ \underline{x} \in \mathbb{R}^n \mid A\underline{x} = \underline{0} \}$$

UNDERDETERMINED SYSTEMS (LANDSCAPE)

$\boxed{2}$

Example: Find $\mathcal{C}(A)$, $\mathcal{N}(A)$, and solve it.

$$A = \begin{bmatrix} \boxed{1} & 4 & 1 & 3 \\ 2 & 8 & 6 & 11 \\ 4 & 16 & -8 & -3 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 2 \\ 8 \\ -4 \end{bmatrix}$$

LU

$$\begin{matrix} 1 & 4 & 1 & 3 \\ 0 & 0 & \boxed{4} & 5 \\ 0 & 0 & -12 & -15 \end{matrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} \boxed{1} & 4 & 1 & 3 \\ 0 & 0 & \boxed{4} & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathcal{C}(A)$

$$\text{basis}\left(\mathcal{C}(A)\right) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ -8 \end{bmatrix} \right\}$$

$\boxed{3}$

Null space

$A\underline{x} = \underline{0}$ $\qquad$ $L\underline{c} = \underline{0}$, $\underline{c} = \underline{0}$

$U\underline{x} = \underline{0}$ $\qquad$ $U\underline{x} = \underline{0}$

$$\begin{array}{cccc} \left[\begin{matrix} \boxed{1} & 4 & 1 & 3 \\ 0 & 0 & \boxed{4} & 5 \\ 0 & 0 & 0 & 0 \end{matrix}\right] \\ \begin{matrix} x_1 & \uparrow & x_3 & \uparrow \\ & \lambda & & \mu \end{matrix} \end{array} \begin{bmatrix} x_1 \\ \lambda \\ x_3 \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_3 + 5\mu = 0 \, , \quad x_3 = -\tfrac{5}{4}\mu$$

$$x_1 + 4\lambda + \left(-\tfrac{5}{4}\mu\right) + 3\mu = 0$$

$$x_1 + 4\lambda + \tfrac{7}{4}\mu = 0 \, , \quad x_1 = -4\lambda - \tfrac{7}{4}\mu$$

$\boxed{4}$

$$\underline{x} = \begin{bmatrix} -4\lambda - \tfrac{7}{4}\mu \\ \lambda \\ -5/4\,\mu \\ \mu \end{bmatrix} = \lambda \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} -7/4 \\ 0 \\ -5/4 \\ 1 \end{bmatrix}$$

$$\text{basis}(N(A)) = \left\{ \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -7/4 \\ 0 \\ -5/4 \\ 1 \end{pmatrix} \right\}$$

Solve

$L\underline{c} = \underline{b}$, $U\underline{x} = \underline{c}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ -4 \end{bmatrix} \qquad \underline{c} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

$$Ux = c.$$

$$\begin{array}{cccc} \\ \end{array}\begin{bmatrix} \boxed{1} & 4 & 1 & 3 \\ 0 & 0 & \boxed{4} & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ \lambda \\ x_3 \\ \mu \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

$$\begin{array}{cccc} & \uparrow & & \uparrow \\ & \lambda & & \mu \end{array}$$

$$4x_3 + 5\mu = 4 \quad , \quad x_3 = 1 - \frac{5}{4}\mu$$

$$x_1 + 4\lambda + \left(1 - \frac{5}{4}\mu\right) + 3\mu = 2$$

$$x_1 = 1 - 4\lambda - \frac{7}{4}\mu$$

$$\underline{x} = \begin{bmatrix} 1 - 4\lambda - \frac{7}{4}\mu \\ \lambda \\ 1 - \frac{5}{4}\mu \\ \mu \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\underline{x}_p} + \underbrace{\lambda \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} -7/4 \\ 0 \\ -5/4 \\ 1 \end{bmatrix}}_{\underline{x}_n}$$

$\underline{x}_p$: particular solution.

$\underline{x}_n$: null space solution.

$$A\underline{x} = A(\underline{x}_p + \underline{x}_n)$$

$$= A\underline{x}_p + \underbrace{\cancel{A\underline{x}_n}}_{\underline{0}} = A\underline{x}_p$$

AN APPLICATION FOR UND. SYSTEMS

Fit a curve through points.

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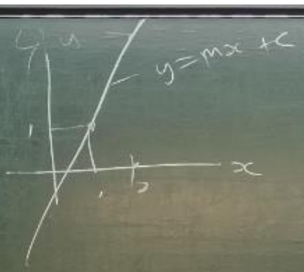
<u>Line:</u> $(1,1), \ (4,2)$

$$y = mx + c$$

$$1 = m \cdot 1 + c$$

$$2 = m4 + c$$

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

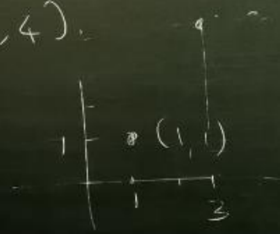


Find parabola of the form $y = ax^2 + bx + c$ through $(1,1)$ and $(3,4)$.

$$y = ax^2 + bx + c$$

$$1 = a1 + b1 + c$$

$$4 = a9 + b3 + c$$



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$$\begin{bmatrix} 1 & 1 & 1 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -6 & -8 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 \\ 9 & 1 \end{bmatrix}$$

$Lc = b$

$$\begin{bmatrix} 1 & 0 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad c_1 = 1$$

$$9 \cdot 1 + c_2 = 4, \quad c_2 = -5$$

$$c = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$Ux = c$

$$\begin{array}{c} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{-6} & -8 \end{bmatrix} \\ \qquad\qquad \uparrow \\ \qquad\qquad \lambda \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$\boxed{9}$

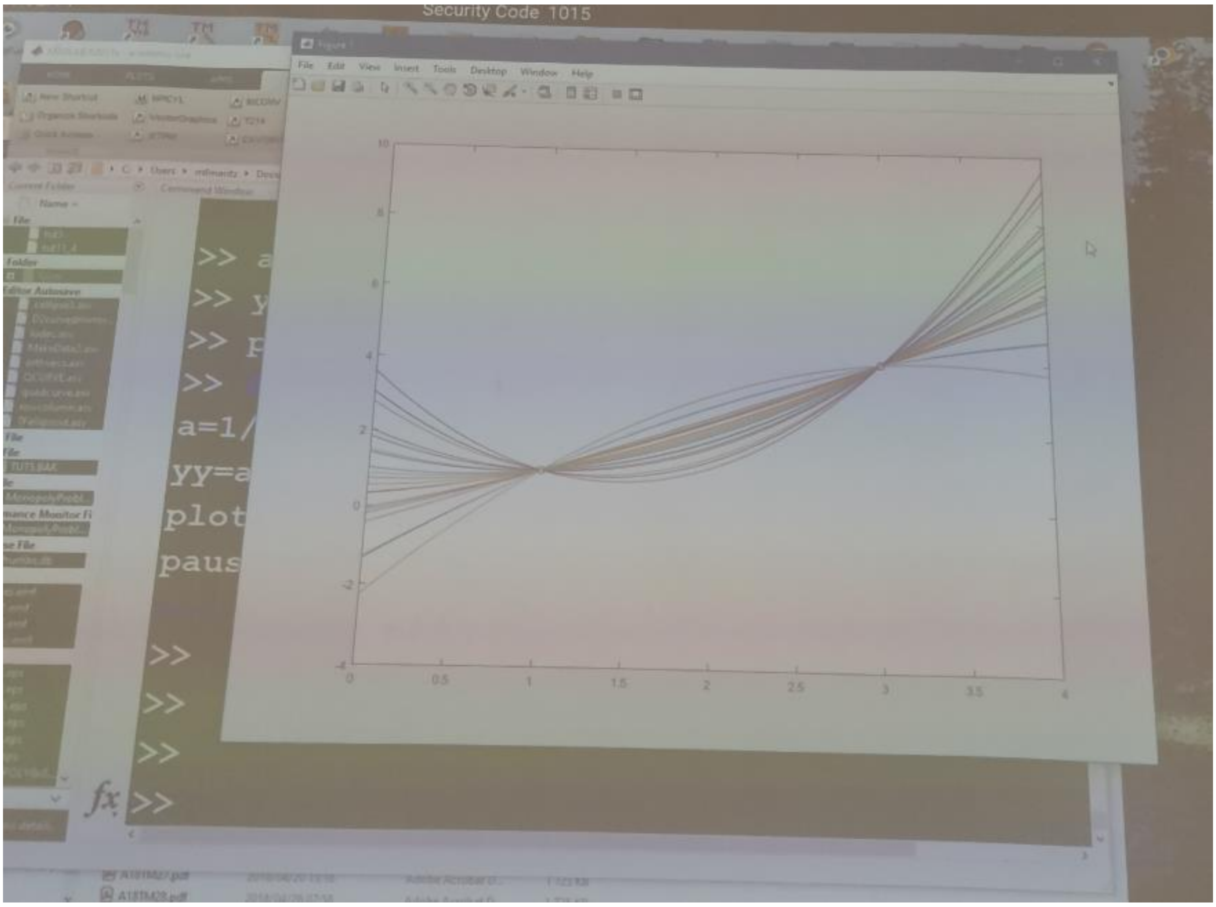
$$+6x_2 + 8\lambda = +5, \quad x_2 = \frac{5}{6} - \frac{4}{3}\lambda$$

$$x_1 + \frac{5}{6} - \frac{4}{3}\lambda + \lambda = 1, \quad x_1 = \frac{1}{6} + \frac{1}{3}\lambda$$

$$-\frac{1}{3}\lambda$$

$$\underline{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{6} + \frac{1}{3}\lambda \\ \frac{5}{6} - \frac{4}{3}\lambda \\ \lambda \end{bmatrix}$$

$$c = \lambda$$
$$b = \frac{5}{6} - \frac{4}{3}\lambda$$
$$a = \frac{1}{6} + \frac{1}{3}\lambda$$



```
>> xx=0:0.1:4;
>> plot([1 3],[1 4],'ro')
>> hold on
>> for k=1:20;
la=randn(1);
a=1/6 + 1/3*la;  b=5/6 -4/3*la; c=la;
yy=a*xx.^2+ b*xx + c;
plot(xx,yy)
pause
end
>>
```