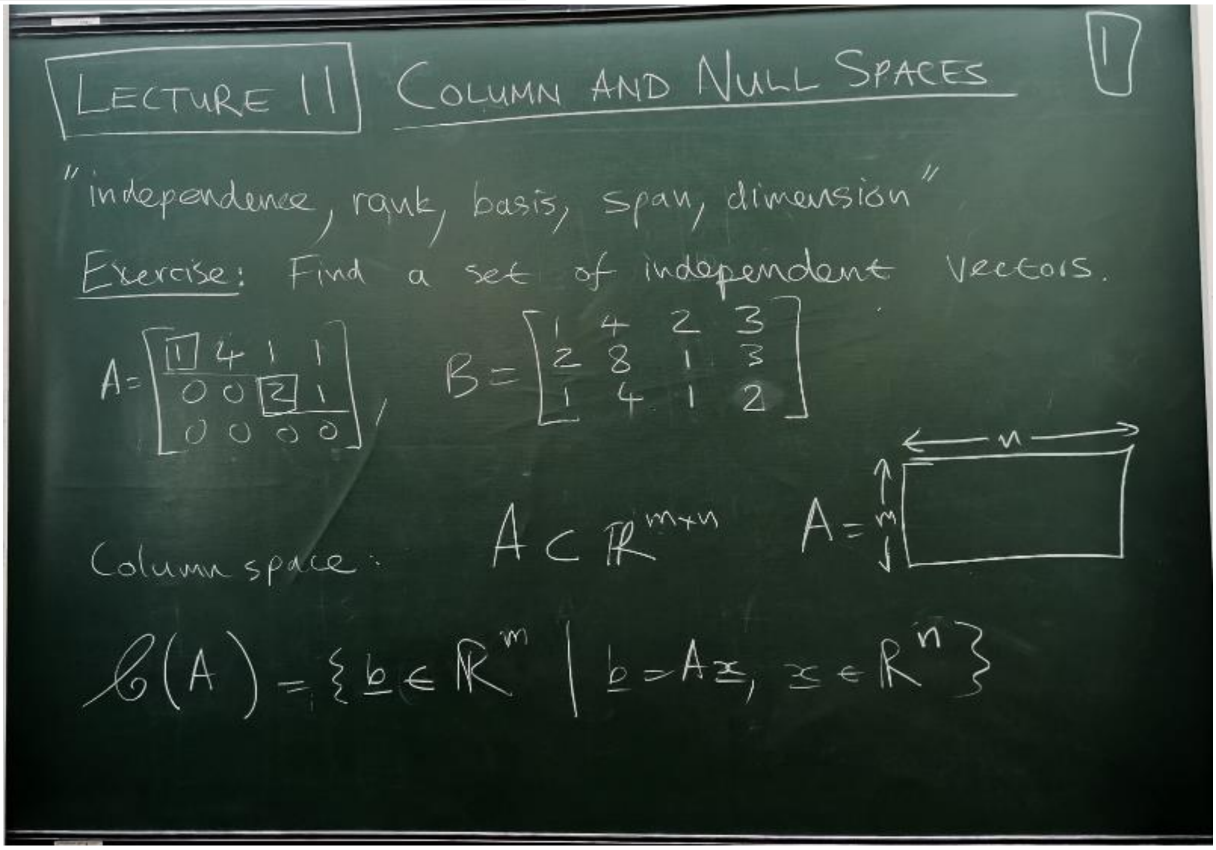
AM214-2023: LECTURE 11



LECTURE 11 COLUMN AND NULL SPACES

$\boxed{1}$

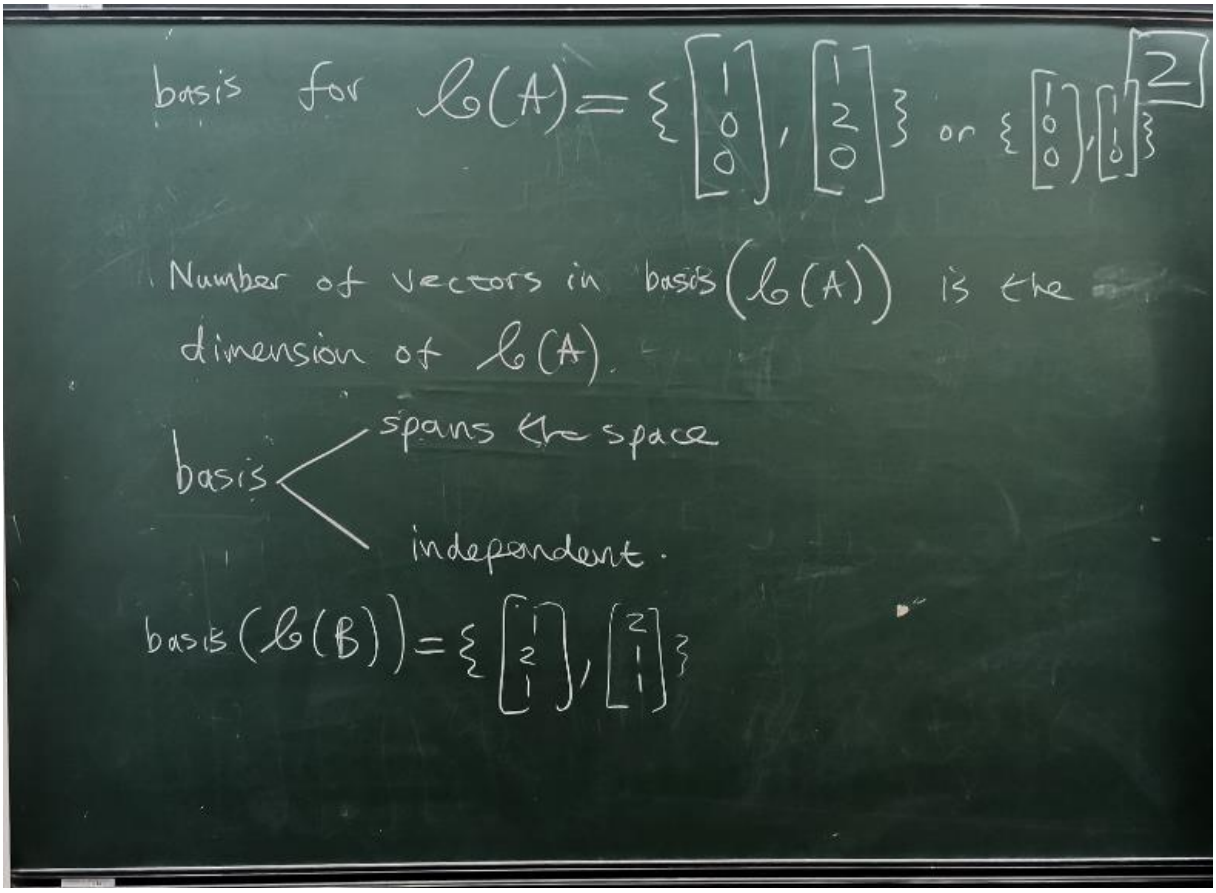
"independence, rank, basis, span, dimension"

Exercise: Find a set of independent vectors.

$$A = \begin{bmatrix} \boxed{1} & 4 & 1 & 1 \\ 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 2 & 8 & 1 & 3 \\ 1 & 4 & 1 & 2 \end{bmatrix}$$

Column space: $A \subset \mathbb{R}^{m \times n}$ $A =$

$$\mathcal{C}(A) = \{ \underline{b} \in \mathbb{R}^m \mid \underline{b} = A\underline{x}, \ \underline{x} \in \mathbb{R}^n \}$$



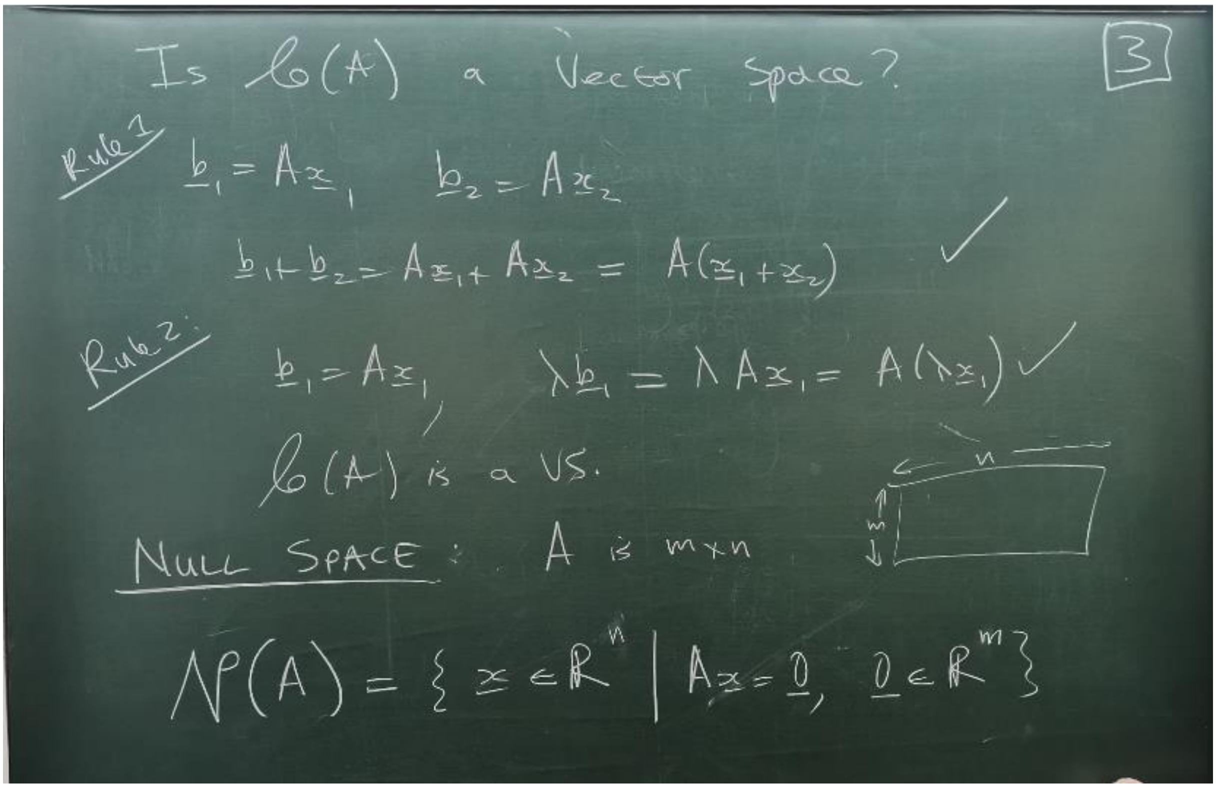
$\boxed{2}$

basis for $\mathcal{C}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

Number of vectors in basis$(\mathcal{C}(A))$ is the dimension of $\mathcal{C}(A)$.

basis
- spans the space
- independent.

$$\text{basis}(\mathcal{C}(B)) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$



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Is $\mathcal{C}(A)$ a Vector Space?

Rule 1 $\underline{b}_1 = A\underline{x}_1 \quad \underline{b}_2 = A\underline{x}_2$

$$\underline{b}_1 + \underline{b}_2 = A\underline{x}_1 + A\underline{x}_2 = A(\underline{x}_1 + \underline{x}_2) \checkmark$$

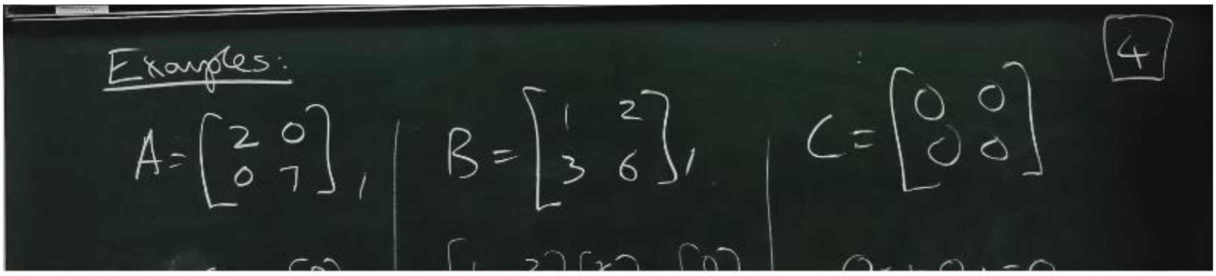
Rule 2: $\underline{b}_1 = A\underline{x}_1, \quad \lambda\underline{b}_1 = \lambda A\underline{x}_1 = A(\lambda\underline{x}_1) \checkmark$

$\mathcal{C}(A)$ is a VS.

NULL SPACE: A is m×n

$$\mathcal{N}(A) = \{\underline{x} \in \mathbb{R}^n \mid A\underline{x} = \underline{0},\ \underline{0} \in \mathbb{R}^m\}$$

ONLY PART OF SLIDE 4 WAS CAPTURED.



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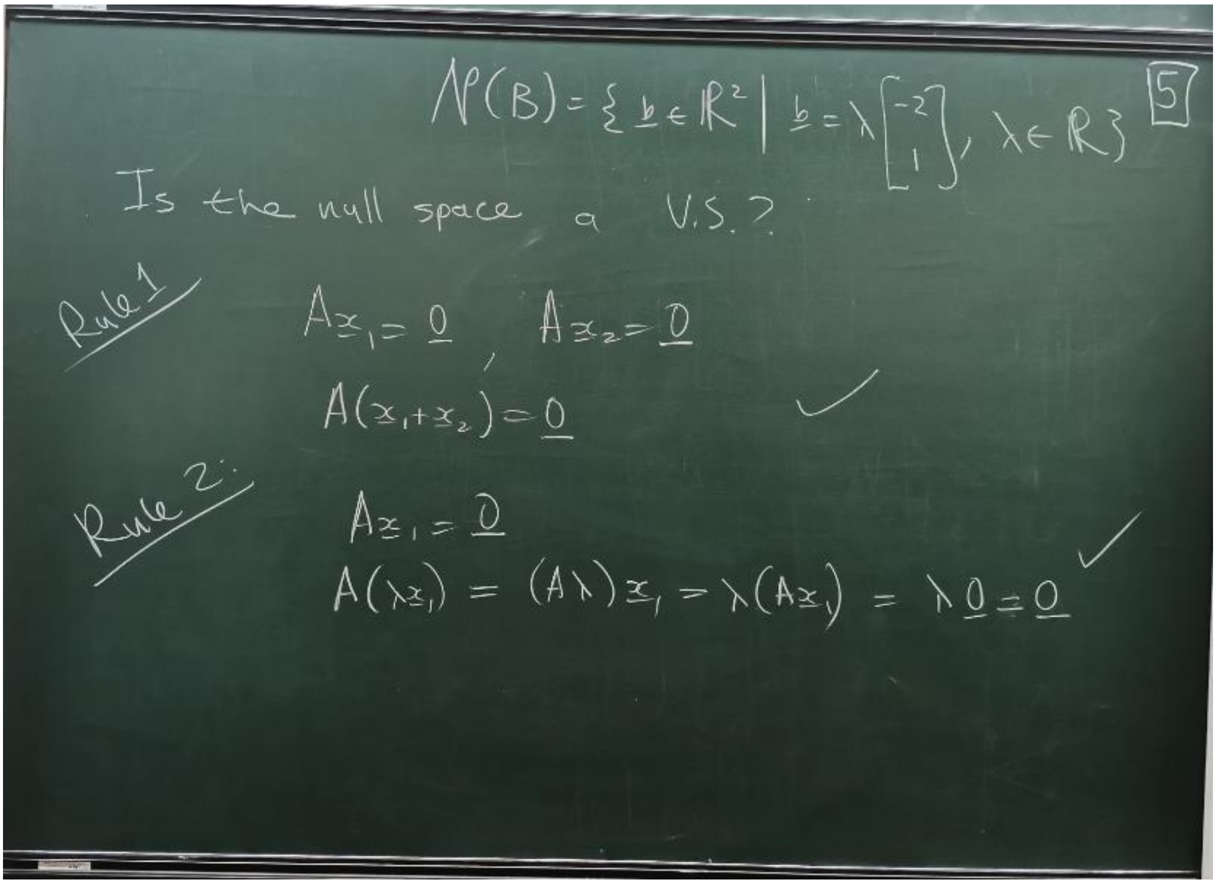
Examples:

$A = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$



$$\mathcal{N}(B) = \{ \underline{b} \in \mathbb{R}^2 \mid \underline{b} = \lambda \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \lambda \in \mathbb{R} \}$$ $\boxed{5}$

Is the null space a V.S.?

Rule 1

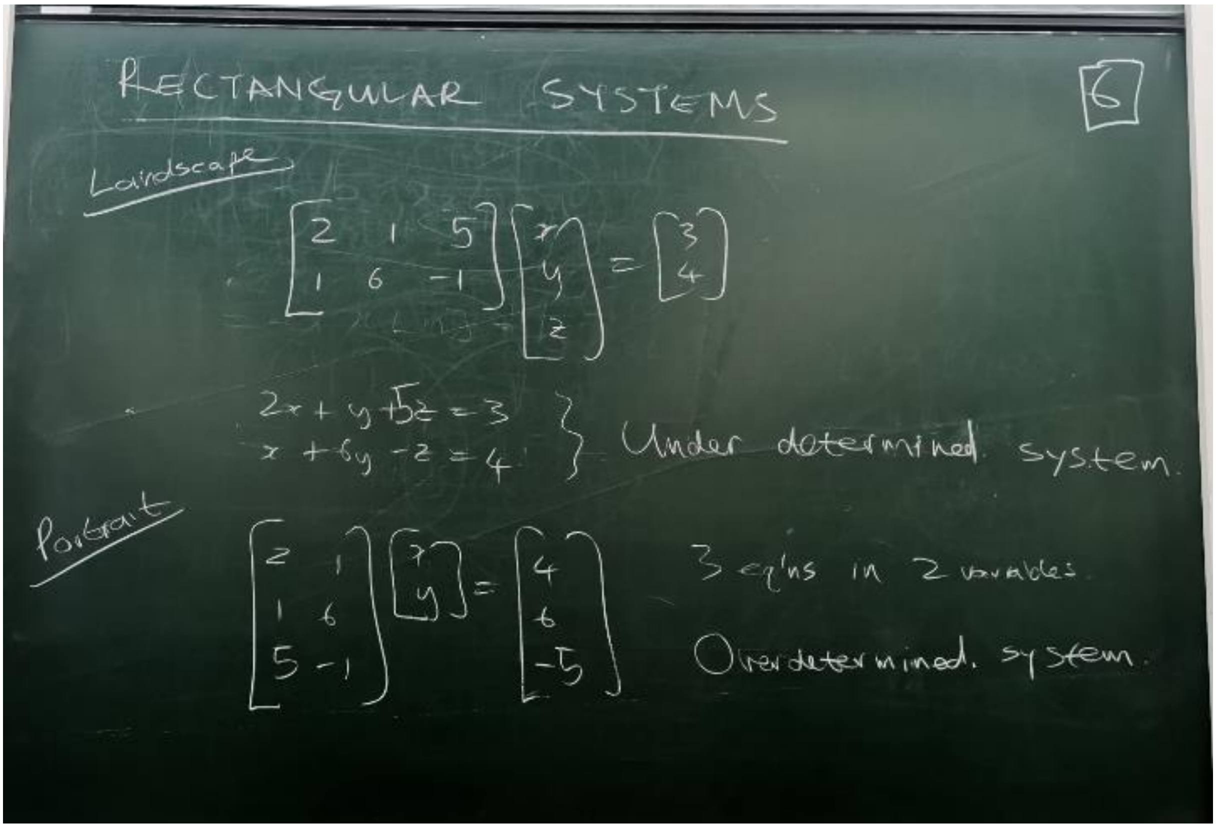
$A\underline{x}_1 = \underline{0}$, $A\underline{x}_2 = \underline{0}$

$A(\underline{x}_1 + \underline{x}_2) = \underline{0}$ ✓

Rule 2:

$A\underline{x}_1 = \underline{0}$

$A(\lambda \underline{x}_1) = (A\lambda)\underline{x}_1 = \lambda(A\underline{x}_1) = \lambda \underline{0} = \underline{0}$ ✓



RECTANGULAR SYSTEMS

$\boxed{6}$

Landscape

$$\begin{bmatrix} 2 & 1 & 5 \\ 1 & 6 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

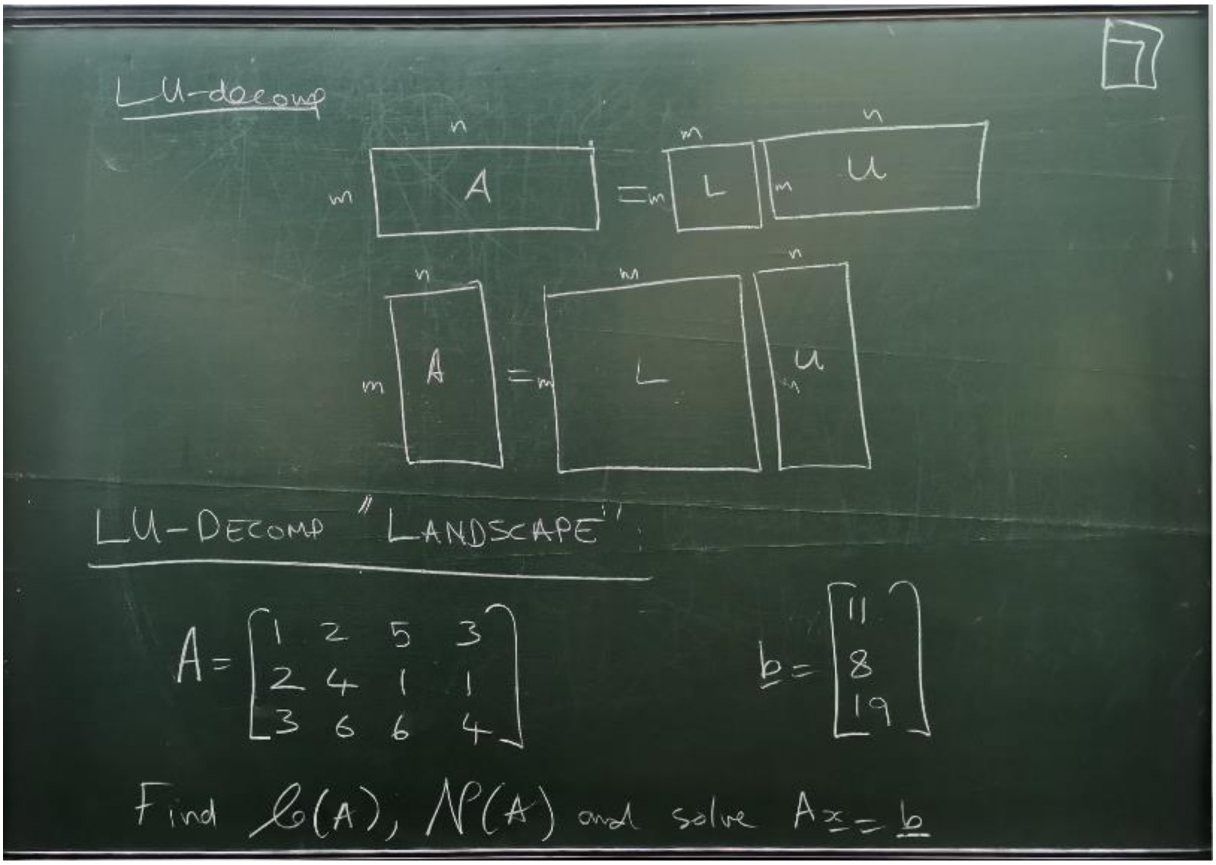
$$\left.\begin{aligned} 2x + y + 5z &= 3 \\ x + 6y - z &= 4 \end{aligned}\right\}$$ Under determined system.

Portrait

$$\begin{bmatrix} 2 & 1 \\ 1 & 6 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -5 \end{bmatrix}$$

3 eq'ns in 2 variables

Overdetermined. system.



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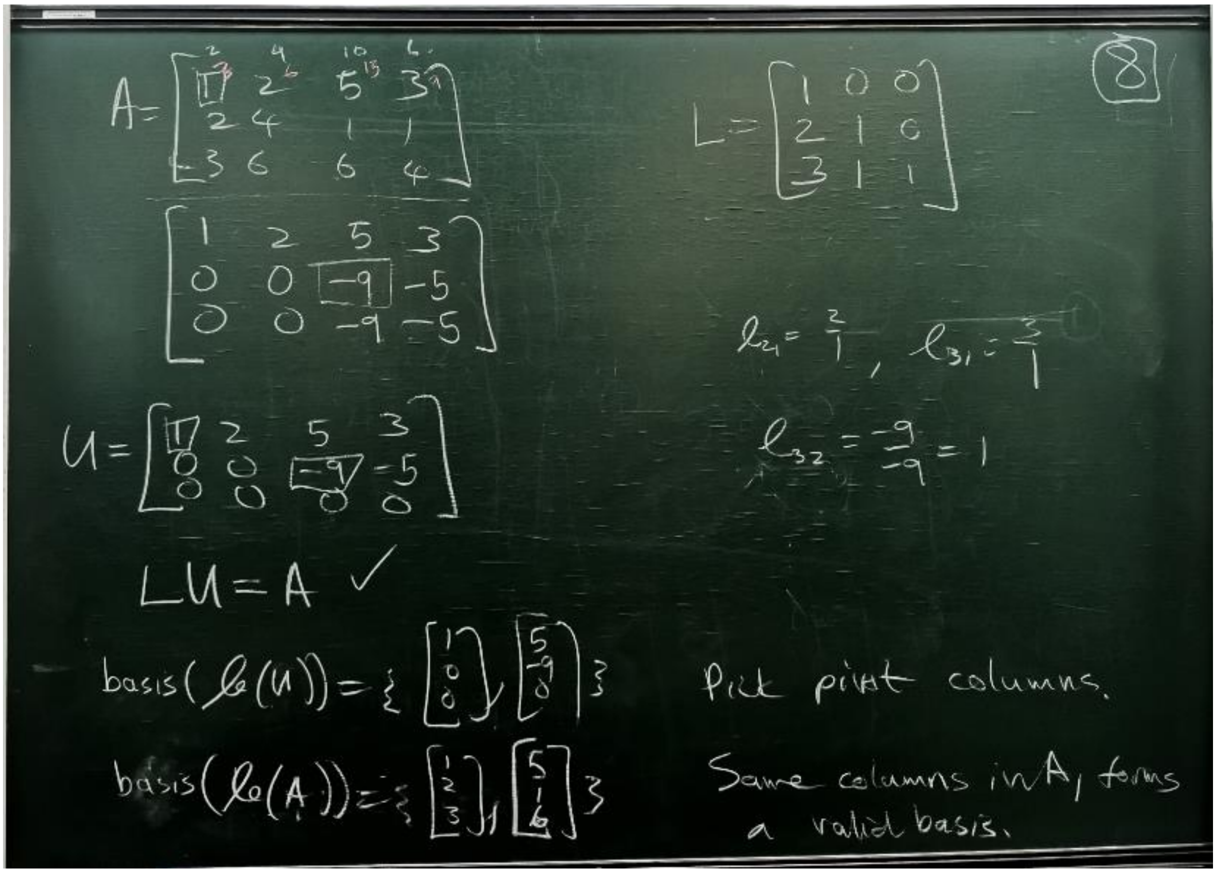
LU-decomp

LU-DECOMP "LANDSCAPE"

$$A = \begin{bmatrix} 1 & 2 & 5 & 3 \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 6 & 4 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 11 \\ 8 \\ 19 \end{bmatrix}$$

Find $\mathcal{C}(A)$, $\mathcal{N}(A)$ and solve $A\underline{x} = \underline{b}$



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$$A = \begin{bmatrix} \overset{2}{\boxed{1}^{3}} & \overset{4}{2^{6}} & \overset{10}{5^{15}} & \overset{6}{3^{9}} \\ 2 & 4 & 1 & 1 \\ 3 & 6 & 6 & 4 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 5 & 3 \\ 0 & 0 & \boxed{-9} & -5 \\ 0 & 0 & -9 & -5 \end{bmatrix}$$

$l_{21} = \frac{2}{1}$, $l_{31} = \frac{3}{1}$

$l_{32} = \frac{-9}{-9} = 1$

$$U = \begin{bmatrix} \boxed{1} & 2 & 5 & 3 \\ 0 & 0 & \boxed{-9} & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$LU = A$ ✓

$\text{basis}(\mathcal{C}(U)) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -9 \\ 0 \end{bmatrix} \right\}$ Pick pivot columns.

$\text{basis}(\mathcal{C}(A)) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} \right\}$ Same columns in A, forms a valid basis.