AM214-2023: LECTURE 10

LECTURE 10 | VECTOR SPACES + LINEAR INDEPENDENCE

1

$$A = \left\{ \underline{a} \in \mathbb{R}^3 \;\middle|\; \underline{a} = \lambda \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ 0 \\ 10 \end{bmatrix},\ \lambda, \mu \in \mathbb{R} \right\}$$

$$B = \left\{ \underline{b} \in \mathbb{R}^3 \;\middle|\; \underline{b} = \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \kappa \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix},\ \kappa, \lambda, \mu \in \mathbb{R} \right\}$$

"span", "dimension", "rank", "independence"

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 10 \end{bmatrix} \right\}$ spans A. A is a line

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$ spans B. B is a plane.

2

Dimension of a line is 1

plane is 2

Informal defn

A set of vectors $\{\underline{a}_1, \underline{a}_2 \ldots \underline{a}_n\}$ is linearly independent if none of them is a combination of any of the others.

Example.

$P = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ space spanned P is a line.

$\underline{0}$ is never independent.

$Q = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ space spanned by Q is a plane.

$\boxed{3}$

$R = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ space spanned by R is a point.

R is dependent.

Number of vectors		
1	dimension = 1 	$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ dimension = 0 
2	dim = 2 	dim = 1 

$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\boxed{4}$

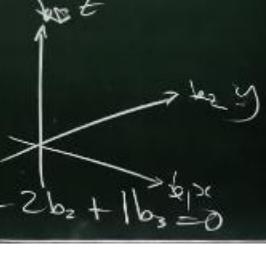
3	dim = 3 	dim = 2 	dim = 1 

V.S's IN TERMS OF EQUATIONS

Find basis for:

$$C = \{ \underline{b} \in \mathbb{R}^3 \mid b_1 - 2b_2 + b_3 = 0 \}$$

$$b_3 = -b_1 + 2b_2$$



$1b_1 - 2b_2 + 1b_3 = 0$

$$1x - 2y + 1z = 0 \qquad \boxed{5}$$

$$\underline{u} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ -b_1 + 2b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ -b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 2b_2 \end{bmatrix}$$

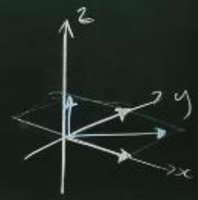
$$= b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

basis for C is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

$E = \mathbb{R}^3$

(xy-plane) basis for $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$= \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$



$\boxed{6}$

$$D = \{ \underline{d} \in \mathbb{R}^3 \mid d_1 + d_2 = 0, \; d_3 = d_1 + d_2 \}$$

$$E = \{ \underline{e} \in \mathbb{R}^3 \mid e_1 + e_2 = 1 \}$$

$d_2 = -d_1$

$d_3 = d_1 + d_2$

$$\underline{d} = \begin{bmatrix} d_1 \\ -d_1 \\ d_1 - d_1 \end{bmatrix} = \begin{bmatrix} d_1 \\ -d_1 \\ 0 \end{bmatrix} = d_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

basis for D is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

$$0x + 0y + 0z = 0$$

$\boxed{7}$

Definition: Independence:

$\{\underline{a}_1, \underline{a}_2 \cdots \underline{a}_n\}$ is independent if, we have

$$\lambda_1 \underline{a}_1 + \lambda_2 \underline{a}_2 + \lambda_3 \underline{a}_3 + \cdots \lambda_n \underline{a}_n = \underline{0} \quad \text{———} \; (1)$$

and the only solution to (1) is, $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$, $\vdots$, $\lambda_n = 0$

Example: $\{\underline{a}_1, \underline{a}_2, \underline{a}_3\}$

$$\underline{a}_3 = \underline{a}_2 + 5\underline{a}_1$$

$$5\underline{a}_1 + 1\underline{a}_2 - 1\underline{a}_3 = \underline{0}$$

$\boxed{8}$

Definition: Independence using a matrix:

The columns of A are independent, if the solution of $A\underline{x} = \underline{0}$, is only $\underline{x} = \underline{0}$

$$\begin{bmatrix} | & | & \\ \underline{a}_1 & \underline{a}_2 & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} | \\ \underline{a}_1 \\ | \end{bmatrix} x_1 + \begin{bmatrix} | \\ \underline{a}_2 \\ | \end{bmatrix} x_2 \cdots = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$