AM214-2023: LECTURE 8

$\boxed{\text{LECTURE 8}}$ LU-DECOMP. (EXAMPLES) $\boxed{1}$

Revise: Two properties of E_{jk}:

Inverse

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ l & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -l & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Order

$$E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$\xrightarrow{\hspace{2em}}$ decreasing indice

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{32}l_{21}+l_{31} & l_{32} & 1 \end{bmatrix}$$

$l_{32}l_{21}+l_{31}$

$$E_{21}E_{31}E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix}}$$ $\boxed{2}$

$\xrightarrow{\hspace{2em}}$ increasing indices

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

LU-DECOMP:

$$A = \begin{bmatrix} \boxed{\times} & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}$$

$$E_{31}^{-1}E_{21}^{-1}A = \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}$$

$$E_{32}^{-1}E_{31}^{-1}E_{21}^{-1}A = \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}$$

$$E_{21}^{-1}A = \begin{bmatrix} 1 & 0 & 0 \\ l & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} a & b & c \\ al+d & bl+e & cl+f \\ g & h & i \end{bmatrix}$$

$al+d=0$

$l = \frac{-d}{a}$ for E_{21}^{-1}

$\ell = \frac{d}{a}$ for E_{ij}^{-1}

$$\underbrace{L^{-1}}\ = U$$

$$\underbrace{(E_{21}E_{31}E_{32})^{-1}}_{L} A = U$$

$$\boxed{A = LU}$$

$\ell = \frac{d}{a}$ for E_{21} $\boxed{3}$

$\ell = \frac{\text{leading coef}}{\text{pivot}}$ (where you want a 0)

<u>Example:</u> Solve $A\underline{x} = \underline{b}$ ~ LU $\boxed{4}$

$$A = \begin{array}{ccc} \cancel{9} & \cancel{3} & \cancel{15} \end{array}$$
$$A = \begin{bmatrix} \boxed{3} & 1 & 5 \\ -9 & -1 & -6 \\ -3 & 7 & 20 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 14 \\ -27 \\ 35 \end{bmatrix}$$

$$\cancel{E_{31}^{-1}E_{21}^{-1}}A = \begin{bmatrix} 3 & 1 & 5 \\ 0 & \boxed{2}^{-8} & 9^{-36} \\ 0 & 8 & 25 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}$$

$$\cancel{E \cdots} A = \begin{bmatrix} 3 & 1 & 5 \\ 0 & 2 & 9 \\ 0 & 0 & -11 \end{bmatrix} = U$$

Check LU

$$LU = A \checkmark$$

$$A\underline{x} = \underline{b}$$
$$L\underbrace{U\underline{x}}_{\underline{c}} = \underline{b}$$
$$\overset{\checkmark}{L}\,\overset{?}{\underline{c}} = \overset{\checkmark}{\underline{b}} \leftarrow \text{forw. subs.}$$
$$\overset{\checkmark}{U}\,\overset{?}{\underline{x}} = \overset{\checkmark}{\underline{c}} \leftarrow \text{backw. subs.}$$

$\boxed{5}$

$L\underline{c} = \underline{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -27 \\ 35 \end{bmatrix}$$

$c_1 = 14.$

$(-3)(14) + c_2 = -27$

$c_2 = 15$

$-(14) + 4(15) + c_3 = 35, \quad c_3 = -11$

$U\underline{x} = \underline{c}.$

$$\underline{c} = \begin{bmatrix} 14 \\ 15 \\ -11 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 5 \\ 0 & 2 & 9 \\ 0 & 0 & -11 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 15 \\ -11 \end{bmatrix}$$

$z = 1$

$2y + 9(1) = 15, \quad y = 3.$

$3x + 1(3) + 5(1) = 14, \quad x = 2$

$$\underline{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

Check:

$$A\underline{x} = \begin{bmatrix} \; \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \; \end{bmatrix}$$

What if a pivot is zero?

$\boxed{6}$

$$B = \begin{bmatrix} 0 & 2 & 9 \\ 3 & -5 & -22 \\ 12 & 2 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PB = \begin{bmatrix} 3 & -5 & -22 \\ 0 & 2 & 9 \\ 12 & 2 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & & \end{bmatrix}$$

$$p = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \qquad p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$PB = LU$

$B\underline{x} = \underline{b}$

$(PB)\underline{x} = P\,\underline{b}$

$$A\underline{x} = \underline{b}$$

$$AI\underline{x} = \underline{b}$$

$$A(\cancel{P^{-1}P})\underline{x} = \underline{b}$$

$$\cancel{(AP^{-1})(P\underline{x})} = \underline{b}$$

$\boxed{7}$

A is singular

$$U = \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \times & 1 & 0 \\ \times & \times & 1 \end{bmatrix}$$

$L\underline{c} = \underline{b}$ ← always a solution

$$U\underline{x} = \underline{c}$$

$$\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ ? \end{bmatrix}$$

non zero

$0x + 0y + 0z =$ non-zero.