AM214-2023: LECTURE 6

LECTURE 6 — LU-DECOMPOSITION

$\boxed{1}$

$$2x + 2y - 5z = -6$$
$$6x + 9y - 14z = -13$$
$$-4x + 11y + 10z = 27$$

$$\underbrace{\begin{bmatrix} 2 & 2 & -5 \\ 6 & 9 & -14 \\ -4 & 11 & 10 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} -6 \\ -13 \\ 27 \end{bmatrix}}_{\underline{b}}$$

$$A \overset{?}{\underline{x}} = \underline{b}$$

$\boxed{2}$

$$A = LU$$

lower triangular, upper

Solve system with LU.

$$A\underline{x} = \underline{b}$$

$$L(\underbrace{U\underline{x}}_{\underline{c}}) = \underline{b} \longrightarrow L\overset{?}{\underline{c}} = \underline{b}$$

Forward substitution

$$\begin{bmatrix} 4 & 0 & 0 \\ 5 & 7 & 0 \\ 6 & 8 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$U\underline{x} = \underline{c}$$

$$\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$c_1 = \frac{b_1}{4}$$

$$5c_1 + 7c_2 = b_2 \longrightarrow c_2$$

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→ Backword substitution

$$E(E, A) = \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix} = U$$

Elementary row-op matrices

Replace row 2 with row 2 + ℓ(row 1)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{this}]{\text{Do}} \begin{bmatrix} 1 & 0 & 0 \\ \ell & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_{21}$$

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$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \ell & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}} \underbrace{\begin{bmatrix} \boxed{a} & b & c \\ d & \boxed{e} & f \\ g & h & \boxed{i} \end{bmatrix}}_{A} = \begin{bmatrix} a & b & c \\ \ell a + d & \ell b + e & \ell c + f \\ g & h & i \end{bmatrix}$$

$$E_{21} A = \begin{bmatrix} a & b & c \\ 0 & \times & \times \\ g & h & i \end{bmatrix}$$

$= 0$, in other words

$$\ell a + d = 0$$

$$\ell = -\frac{d}{a}$$

leading coefficient

multiplier

pivot element

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Inverse of a row-op matrix

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \ell & 0 & 1 \end{bmatrix} \qquad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\ell & 0 & 1 \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} \boxed{2} & 2 & -5 \\ 6 & \boxed{9} & -14 \\ -4 & 11 & \boxed{10} \end{bmatrix}$$

$$(E_{21})^{-1} A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -5 \\ 6 & 9 & -14 \\ -4 & 11 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & -5 \\ 0 & 3 & 1 \\ -4 & 11 & 10 \end{bmatrix}$$

$$\ell_{21} = +\frac{6}{2} = +3$$

$$\ell = \frac{\text{leading c}}{\text{pivot}}$$

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$$(E_{31})^{-1}\left[(E_{21})^{-1} A\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -5 \\ 0 & 3 & 1 \\ -4 & 11 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & -5 \\ 0 & \boxed{3} & 1 \\ 0 & 15 & 0 \end{bmatrix}$$

$$\ell_{31} = \frac{-4}{2} = -2$$

$$(E_{32})^{-1} \downarrow \quad = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -5 \\ 0 & 3 & 1 \\ 0 & 15 & 0 \end{bmatrix}$$

$$\ell_{32} = \frac{15}{3} = 5$$

$$= \begin{bmatrix} 2 & 2 & -5 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix} = U$$

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$$\underbrace{\left(E_{32}^{-1} E_{31}^{-1} E_{21}^{-1}\right)}_{L^{-1}} A = U$$

Call it L^{-1}

$$L^{-1} = E_{32}^{-1} E_{31}^{-1} E_{21}^{-1}$$

$$L = \left(E_{32}^{-1} E_{31}^{-1} E_{21}^{-1}\right)^{-1}$$

$$= E_{21} E_{31} E_{32}$$

$$E_{21} E_{31} E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ +3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +5 & 1 \end{bmatrix}$$

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$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 5 & 1 \end{bmatrix}$$

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -5 \\ 0 & 3 & 1 \\ 0 & 0 & -5 \end{bmatrix} = A$$