AM214-2023: LECTURE 5

1

LECTURE 5 — SPECIAL SQUARE MATRICES

$A \in \mathbb{R}^{n \times n}$

Solving matrix equations: Solve X

① $A^{-1}(X-I)A + (X-A^{-1})B^TA = A - I + XB^TA$

$$A^{-1}XA - \cancel{I} + \cancel{XB^TA} - A^{-1}B^TA - \cancel{XB^TA} = A - \cancel{I}$$

$$A^{-1}XA = A + A^{-1}B^TA$$

Premult. by A: $XA = A^2 + B^TA$

Postmult by A^{-1}: $\boxed{X = A + B^T}$

2

② $B^TX(B^T+I)^{-1} - B^TB(B^T+I)^{-1} = B^T$

Postmult. by (B^T+I):

$$B^TX - B^TB = B^T(B^T+I)$$

Premult. by $(B^T)^{-1}$: $X - B = B^T + I$

$$\boxed{X = B + B^T + I}$$

$(AB^T)^{-1}$

$(B^T)^{-1}A^{-1}$

Symmetry:

3

S is symmetric, if $S^T = S$

$$S = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

A is antisymmetric if $A^T = -A$

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

Any $B \in \mathbb{R}^{n \times n}$, can be written as $B = S + A$

4

$B = S + A$ — ① ($S^T = S$, $A = -A^T$)

Transpose: $B^T = (S+A)^T = S^T + A^T = S - A$

$B^T = S - A$ — ②

① + ②: $B + B^T = 2S$ $S = \dfrac{B + B^T}{2} = \frac{1}{2}(B + B^T)$

① − ②: $B - B^T = S + A - (S - A)$

$= 2A$ $A = \frac{1}{2}(B - B^T)$

Triangular mat's:

$L =$ Lower triangular $= \begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}$

Upp... $= \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$

5

$L_1 L_2$

Diagonal matrices:

$$D = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

$$D^2 = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \gamma^2 \end{bmatrix}$$

$$D^m = \begin{bmatrix} \alpha^m & 0 & 0 \\ 0 & \beta^m & 0 \\ 0 & 0 & \gamma^m \end{bmatrix} \qquad D^{-1} = \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\beta} & 0 \\ 0 & 0 & \frac{1}{\gamma} \end{bmatrix}$$

$A^2 = AA$, $\quad A^{\frac{1}{2}}$ $\qquad A = SDS^{-1}$

$$A^m = SD(S^{-1}S)DS^{-1} = SD^mS^{-1}$$

PRE MULTIPLICATION AS AN OPERATOR

6

$$BA = A' \qquad\qquad AB = A'$$

Premult by diag. mat. scales the rows.

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b & \alpha c \\ \beta d & \beta e & \beta f \\ \gamma g & \gamma h & \gamma i \end{bmatrix}$$

Post mult. by a diag. mat. scales columns.

PERMUTATION MAT'S

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{7}$

$$P_{(132)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

$$P_{(312)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^{-1} = P^{T}$$

<u>Elementary row-operation mat's.</u>

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \ell & 0 & 1 \end{bmatrix}$$