

$$1 \quad A^T x = \begin{bmatrix} \alpha x_1 + x_2 + \beta x_3 \\ x_2 + \gamma x_3 \\ \beta x_2 + \alpha x_1 \end{bmatrix} = \begin{bmatrix} \alpha & 1 & \beta \\ 0 & 1 & \gamma \\ \beta & \alpha & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

See the last page.

(b) or (e)

2, 3. (a) One solution

2. (a)

(b) no solution

3. (e)

(c) no solution

(d) no solution

(e) Inf. many sols

$$4. \quad BXA - BA + \cancel{XA} = B^2A + \cancel{XA}$$

$$BXA = BA + B^2A$$

$$\text{Pre-multiply by } A^{-1}: \quad XA = A + BA$$

$$\text{Post-multiply by } A^{-1}: \quad X = I + B$$

(c)

$$5. \quad b = \begin{bmatrix} b_1 + b_2 \\ b_2 \\ b_3 \end{bmatrix} = b_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

2

(c)

$$6. 7. \quad \begin{matrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ & & & & & \textcircled{5} \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}$$

$$\boxed{1} \quad \begin{matrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & & & & & \\ & 4 & & & & \end{matrix}$$

6. \_\_\_\_\_ (b)

7. rank is 3 \_\_\_\_\_ (c)

8.  $P_L = \frac{qq^T}{q^T q}$  projects on line  
spanned by  $q$

(3)

$$P = I - P_L \quad \text{projects on plane } V.$$

$$W = I - 2P = I - 2(I - P_L)$$

$$= I - 2I + 2P_L = 2P_L - I$$

$$W = 2P_L - I \quad \text{reflects through a line.}$$

→ (d)

q.  $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} \rightarrow \text{is a reflection}$$

about

→ (d)

10.

$$y = a \cdot 1 + bx^2$$

(4)

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 4 \\ 1 & 9 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix} \rightarrow (e)$$

$$11. \quad Fx = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

5

(c)

12.

$$\begin{vmatrix} 1-\lambda & -1 & 2 & 3 \\ 2 & 3-\lambda & 3 & 3 \\ 0 & 0 & 3-\lambda & 1 \\ 0 & 0 & 0 & 9-\lambda \end{vmatrix} = \cancel{(1-\lambda)}$$

$$(9-\lambda)(5-\lambda)((3-\lambda)(1-\lambda)+2) = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm \sqrt{4}}{2}$$

$$= \frac{-4 \pm 2i}{2} = -2 \pm i$$

Eigenvalues:  $9, 5, 2+i, 2-i$

(b)

(6)

B.  $\det(J) = 0.$ 

$$\begin{vmatrix} 1 & -3 & 0 \\ -3 & -\frac{1}{2} & -3 \\ 0 & -3 & -3 \end{vmatrix} \begin{vmatrix} 1 & -3 \\ -3 & -\frac{1}{2} \\ 0 & -3 \end{vmatrix}$$

3(6)

$$0 - 9 + 18 + 1 - 0 + 0 = 10.$$

$$\text{tr}(J) = -1.5$$

~~$$\alpha + \beta = -1.5$$~~

~~$$4\alpha\beta = 10$$~~

~~$$2\beta = 2.5$$~~

~~$$\beta = 2.5$$~~

~~$$\alpha + \beta = -6.5$$~~

~~$$-5.5$$~~

~~$$\alpha^2 + 2\alpha + 5.5 = 0$$~~

$$\alpha = \frac{-5.5 \pm \sqrt{(5.5)^2 - 4(2.9)}}{2}$$

~~$$\alpha_1 = -5$$~~

(e)

**Afdeling B (40 punte)**

Doen hierdie afdeling op die vraestel. Jy mag die agterkante van hierdie vraestel-bladsye ook gebruik.

**Section B (40 marks)**

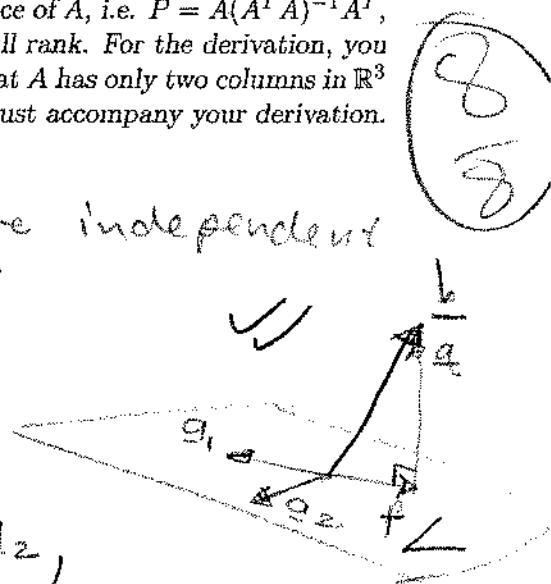
Do this section on the question paper. You may use the reverse sides of the pages also.

**B1** Herlei die vorm van die projeksie-matriks op die kolom-ruimte van  $A$ , naamlik  $P = A(A^T A)^{-1} A^T$ , waar  $A$  vol rang het. Jy mag vir die doel van die herleiding aanvaar dat  $A$  slegs twee kolomme in  $\mathbb{R}^3$  het, en jou herleiding moet van 'n skets vergesel wees.

Derive the form of the projection matrix on [8] the column space of  $A$ , i.e.  $P = A(A^T A)^{-1} A^T$ , where  $A$  has full rank. For the derivation, you may assume that  $A$  has only two columns in  $\mathbb{R}^3$  and a sketch must accompany your derivation.

Let  $A = \begin{bmatrix} 1 & 1 \\ q_1 & q_2 \\ 1 & 1 \end{bmatrix}$ ,  $q_1$  and  $q_2$  are independent and span  $V$

$f$  is the projection of  $b$



on  $V$ . Then  $f = x_1 q_1 + x_2 q_2$ ,

$$\text{or } f = \begin{bmatrix} 1 & 1 \\ q_1 & q_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \underline{x}.$$

$\underline{x}$  is unknown, and needs to be found.

$$\text{Let } p + q = b \text{ or } q = b - f \quad \text{②}$$

$q$  must be orthogonal to both  $q_1$  and  $q_2$ ,

$$\text{or } q^T q = 0, \quad q^T q_1 = 0, \text{ i.e. } A^T q = 0 \quad \text{③}$$

$$\text{Subst. ② into ③: } A^T(b - f) = 0 \quad \text{④}$$

$$\text{Subst. ① into ④: } A^T(b - A\underline{x}) = 0, \text{ i.e.}$$

$$A^T A \underline{x} = A^T b, \text{ or } \underline{x} = (A^T A)^{-1} A^T b \quad \text{where } P = A(A^T A)^{-1} A^T$$

$$\text{then } f = A \underline{x} = A(A^T A)^{-1} A^T b = P b$$

B2 Gebruik die vorm van die refleksiematriks  $H$  hieronder en toon aan dat lig wat vanaf die oorsprong af kom en die reflekterende kromme  $y = \frac{1}{4}x^2 - 1$  tref, na weerkaatsing parallel aan die  $x$ -as sal wees. Maak 'n skets en dui die invallende en gereflekteerde strale duidelik aan.

Wenk: Die refleksiematriks is

Use the form of the reflection matrix  $H$  below and show that light coming from the origin that hits the reflecting curve  $y = \frac{1}{4}x^2 - 1$  will be parallel to the  $x$ -axis after reflection. Also draw a sketch and show the incident and reflected beams of light clearly.

Hint: The reflection matrix is

10

$$\begin{aligned} y &= \left[ \begin{array}{c} x \\ \frac{1}{4}x^2 - 1 \end{array} \right] \\ H &= \frac{1}{1 + f'(x)^2} \begin{bmatrix} 1 - f'(x)^2 & 2f'(x) \\ 2f'(x) & f'(x)^2 - 1 \end{bmatrix} \end{aligned}$$

$$f(x) = \frac{1}{4}x^2 - 1$$

$$f'(x) = \frac{1}{2}x$$

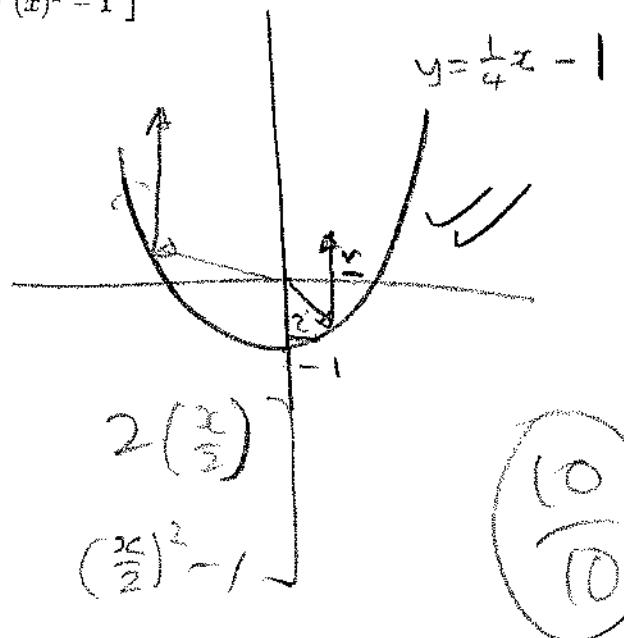
$$H = \frac{1}{1 + (\frac{x}{2})^2}$$

$$\begin{bmatrix} 1 - (\frac{x}{2})^2 & 2(\frac{x}{2}) \\ 2(\frac{x}{2}) & (\frac{x}{2})^2 - 1 \end{bmatrix}$$

$$= \frac{1}{1 + \frac{1}{4}x^2} \begin{bmatrix} 1 - \frac{x^2}{4} & x \\ x & \frac{x^2}{4} - 1 \end{bmatrix}$$

$$\begin{aligned} \underline{v} &= H \underline{v} = \frac{1}{1 + \frac{x^2}{4}} \begin{bmatrix} 1 - \frac{x^2}{4} & x \\ x & \frac{x^2}{4} - 1 \end{bmatrix} \begin{bmatrix} x \\ \frac{x^2}{4} - 1 \end{bmatrix} \\ &= \frac{1}{1 + \frac{x^2}{4}} \begin{bmatrix} x - \frac{x^3}{4} + \frac{x^3}{4} - x \\ ? \end{bmatrix} = \begin{bmatrix} 0 \\ ? \end{bmatrix} \end{aligned}$$

$x$ -component of  $\underline{v}$  is zero,  $\underline{v}$  is parallel to  $y$ -axis.



## USING ERRATA

- B3 Skryf die oorbepaalde stelsel hieronder in die vorm  $Ax = b$ . Doen dan gereduseerde  $\bar{Q}R$ -ontbinding van  $A$  ( $\bar{Q}$  het net twee kolomme) en vind die kleinste-kwadrate oplossing van die stelsel deur van die  $\bar{Q}R$ -ontbinding gebruik te maak.

$$A = \begin{bmatrix} -12 & 0 \\ 6 & 4 \\ 6 & 10 \\ -3 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

$$q^T q = 15^2, \quad q_1 = \frac{1}{15} \begin{bmatrix} -12 \\ 6 \\ 6 \\ -3 \end{bmatrix}$$

$$\begin{aligned} -12x &= 10 \\ 6x + 4y &= 0 \\ 6x + 10y &= 5 \\ -3x + 3y &= 5 \end{aligned}$$

$$\bar{b} = \begin{bmatrix} 10 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -12 \\ 6 \\ 6 \\ -3 \end{bmatrix} - \frac{1}{15} \begin{bmatrix} -12 \\ 6 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

$$R = \begin{bmatrix} 15 & 5 \\ 0 & 10 \end{bmatrix}$$

$$\bar{b}' = b - q_1^T b$$

$$= \begin{bmatrix} 0 \\ 4 \\ 10 \\ 3 \end{bmatrix} - \frac{1}{15} \begin{bmatrix} -4 \\ 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 8 \\ 4 \end{bmatrix}$$

$$q_1^T b = \frac{1}{15} [-4 \ 2 \ 2 \ -1] \begin{bmatrix} 10 \\ 0 \\ 5 \\ 5 \end{bmatrix} = \frac{1}{15} (25) = 5$$

$$\|\bar{b}'\| = 10$$

$$q_2 = \frac{1}{10} \begin{bmatrix} 4 \\ 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 2 \end{bmatrix}$$

$$q_1^T \bar{b}' = \frac{1}{15} [-4 \ 2 \ 2 \ -1] \begin{bmatrix} 10 \\ 0 \\ 5 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2 & 2 & -1 \\ 2 & 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 10 \end{bmatrix}$$

$$\bar{Q} = \frac{1}{5} \begin{bmatrix} -4 & 2 \\ 2 & 1 \\ 2 & 4 \\ -1 & 2 \end{bmatrix}$$

$$R = \begin{bmatrix} 15 & 5 \\ 0 & 10 \end{bmatrix}$$

$$\tilde{x} = \begin{bmatrix} -0.8 \\ 1.0 \end{bmatrix}$$

$$\text{Gelukkig } R\tilde{x} = b$$

$$\begin{bmatrix} 15 & 5 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} -0.8 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$y = 1$$

$$15x + 5 = -7$$

$$15x = -7 - 5 = -12, \quad x = -0.8$$

## CORRECTION

Write the overdetermined system below in [10] the form  $Ax = b$ . Then do reduced  $\bar{Q}R$ -decomposition of  $A$  ( $\bar{Q}$  has only two columns) and find the least squares solution of the system by using the  $\bar{Q}R$ -decomposition.

$$\begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 5 \\ 0 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 4 \\ 10 \\ 10 \end{bmatrix}$$

B3 Skryf die oorbepaalde stelsel hieronder in die vorm  $Ax = b$ . Doe dan gereduseerde  $\bar{Q}R$ -ontbinding van  $A$  ( $\bar{Q}$  het net twee kolomme) en vind die kleinste-kwadrate oplossing van die stelsel deur van die  $\bar{Q}R$ -ontbinding gebruik te maak.

Write the overdetermined system below in [10] the form  $Ax = b$ . Then do reduced  $\bar{Q}R$ -decomposition of  $A$  ( $\bar{Q}$  has only two columns) and find the least squares solution of the system by using the  $\bar{Q}R$ -decomposition.

ALTERNATIVE:

WITHOUT THE  
CORRECTION

$$\begin{array}{lcl} -12x & = & 10 \\ 6x + 4y & = & 0 \\ 6x + 10y & = & 5 \\ -3x - 3y & = & 5 \end{array}$$

$$q^T q = 15 \quad q_1 = \frac{1}{15} \begin{bmatrix} -12 \\ 6 \\ 6 \\ -3 \end{bmatrix} \quad q_1^T b = 6.2$$

$$b' = b - q_1(q_1^T b) = \begin{bmatrix} 0 \\ 4 \\ 10 \\ -3 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} -4 \\ 2 \\ 2 \\ -7 \end{bmatrix} \cdot 6.2$$

$$= \begin{bmatrix} 4.96 \\ 1.52 \\ 7.52 \\ -1.76 \end{bmatrix} \quad \|b'\| = 9.304 \quad q_2 = \begin{bmatrix} 0.5331 \\ 0.1634 \\ 0.9083 \\ -0.1892 \end{bmatrix}$$

$$Q^T b = \begin{bmatrix} -0.8 & 0.4 & 0.4 & -0.2 \\ 0.533 & 0.163 & 0.808 & -0.189 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 8.427 \end{bmatrix}$$

$$R \tilde{x} = Q^T b \quad \begin{bmatrix} 15 & 6.2 \\ 0 & 9.304 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 \\ 8.427 \end{pmatrix} \quad y = 0.9057 \quad x = -0.8410$$

$$\bar{Q} = \begin{array}{c|c} -0.8 & 0.5331 \\ \hline 0.4 & 0.1634 \\ \hline 0.4 & 0.9083 \\ \hline -0.2 & -0.1892 \end{array} \quad R = \begin{array}{c|c} 15 & 6.2 \\ \hline 0 & \uparrow \\ \hline & 9.304 \end{array} \quad \tilde{x} = \begin{array}{c} -0.8410 \\ 0.9057 \end{array}$$

B4 Ontbind die matriks  $A$  hieronder in die vorm  $A = SAS^{-1}$ , waar  $\Lambda$  diagonaal is. Een eievektor van  $A$  word gegee, aangedui met  $x$ , en  $\det(A) = 280$ .

Decompose the matrix  $A$  below in the form  $A = SAS^{-1}$ ,  $\Lambda$  is diagonal. An eigenvector of  $A$  is given below, denoted by  $x$ , and  $\det(A) = 280$ .

12

$$A = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 7 & 2 \\ 0 & 2 & 8 \end{bmatrix}, \quad x = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad Ax = \begin{bmatrix} 14 \\ 7 \\ -14 \end{bmatrix}$$

$$\lambda_1 = 7$$

$$\alpha + \beta + 7 = 14 + 7$$

$$\alpha + \beta = 14$$

$$7\alpha + \beta = 280$$

$$7\alpha + \beta = 40$$

$$\lambda_2 = 4, \quad \lambda_3 = 10.$$

$$\lambda_2 = 4: \quad \left[ \begin{array}{ccc} 2 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 4 \end{array} \right] \rightarrow \left[ \begin{array}{ccc} 2 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{array} \right] \left[ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = 0$$

$$\left[ \begin{array}{c} 2u \\ -3u \\ u \end{array} \right] = \left[ \begin{array}{c} 2 \\ -2 \\ 1 \end{array} \right]$$

$$x_2 + 2u = 0, \quad x_2 = -2u$$

$$2x_1 + 2(-2u) = 0$$

$$2x_1 = 4u, \quad x_1 = 2u$$

$$\lambda_3 = 10: \quad \left[ \begin{array}{ccc} -4 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -2 \end{array} \right] \rightarrow \left[ \begin{array}{ccc} -4 & 2 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{array} \right] \left[ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = 0$$

$$-2x_2 + 2u = 0, \quad x_2 = u$$

$$1 \text{ } 1111111111$$

$$-4(x_1 + 2(u)) = 0$$

$$x_1 = -\frac{1}{2}u$$

S =

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ -2 & 1 & 2 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 7 & & \\ & 4 & \\ & & 10 \end{bmatrix}$$

EINDE VAN TOETS

$$2x_3 = \begin{bmatrix} 2u \\ -3u \\ u \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

END OF TEST

SECTION A, no. 1

$$A^T \begin{pmatrix} x_1 + x_2 + \beta x_3 \\ x_2 + \gamma x_3 \\ \beta x_2 + \alpha x_3 \end{pmatrix} = \begin{bmatrix} \alpha & 1 & \beta \\ 0 & 1 & \gamma \\ 0 & \beta & \alpha \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A = \begin{bmatrix} \alpha & 0 & 0 \\ 1 & 1 & \beta \\ \beta & \gamma & \alpha \end{bmatrix}$$

There is no correct answer.

If the question was: ... then  $A^T = \dots$

then both (b) and (e) are correct.

On the ERRATA page, only (e) is correct.