Department of Mathematical Sciences
Division of Applied Mathematics
Stellenbosch University

2019 | MS763 | Applied Markov Processes

Problems for Chapter 3: Continuous-time Markov chains

Due: 24 September 2019

Theoretical

Q1. (Nuclear physics) There is a particle accelerator between Stellenbosch and Cape Town, called iThemba Labs, used for nuclear research. A typical experiment there involves accelerating particles and smashing them onto a target to see what gets absorbed and what gets emitted back from nuclear reactions. Assume that the probability that the target will absorb r particles over a time Δt is given by a Poisson distribution with parameter a, and that each particle absorbed decays (that is, disappears) in the same time with probability q.

(a) Let X_t be the number of particles absorbed by the target at time t. Write down the transition probability $P(X_{t+\Delta t} = k|X_t = i)$, taking into account absorptions and decays.

(b) Verify that the stationary distribution associated with this Markov chain is a Poisson distribution with parameter a/q, so that

$$P^*(j) = \frac{(a/q)^j}{j!} e^{-a/q}, \quad j = 0, 1, \ldots. \tag{1}$$

(c) What is the expected number of particles absorbed over a long time?

Q2. (Cosmic showers) Particles are created by chain reaction in the atmosphere when high-energy particles reach Earth from outer space. Suppose that one such particle enters the atmosphere and randomly produces with rate λ a new particle having the same reproduction rate. Let $N_t > 0$ be the integer number of particles created at time t starting from $N_0 = 1$.

(a) Write down the master equation for $P_n(t) = P(N_t = n)$.

(b) Find the solution of the master equation starting from $P_n(0) = \delta_{n,1}$. [Hint: Solve the equation for $n = 1$, then for $n = 2$.]

(c) Plot $P_n(t)$ as a function of n for $t \in \{1, 2, 3\}$. Comment on the plot.

(d) What is the expected number of particles as a function of time?

[Note: A proper model of chain reaction should include particle annihilation in addition to creation.]

Q3. (Fish superposition) Flies and wasps land on your fish dinner in the manner of independent Poisson processes with intensities λ and μ, respectively. Show that the aggregate arrivals of flying insects form a Poisson process with intensity $\lambda + \mu$. Give a complete proof.

Q4. (Births with immigration) Let N_t be a birth process with immigration defined by the rate $\lambda_n = \lambda n + \nu$ for the transition $n \to n + 1$, with $\lambda, \nu > 0$.

(a) Write down the master equation for N_t starting from $N_0 = 0$.

(b) Without solving the master equation, show that the expected population $m(t) = E[N_t]$ satisfies the differential equation $\dot{m}(t) = \lambda m(t) + \nu$.

Numerical

Q5. (Comparison of simulation methods) Consider a Poisson process N_t with $\alpha = 0.5$ and $N_0 = 0$.

(a) Simulate and plot 10 trajectories of N_t for $t \in [0, 10]$ using the discretized-time method.

(b) Compute the histogram of N_{10} using the same method. Use enough samples points, that is, enough trajectories. [Note: For this part, you don't need to store whole trajectories.]

(c) Simulate and plot 10 trajectories of N_t for $t \in [0, 10]$ using the random-jump-time method. [Hint: Think of a good way to represent and plot trajectories with this method.]

(d) Compute the histogram of N_{10} using the same method. Use enough samples and don't store whole trajectories.

(e) Compare the two methods.

Q6. (Absorbing state) Consider a birth-death process N_t with birth rate $\alpha = 0.5$, death rate $\beta = 1$, and 0 as an absorbing state so that $W(0 \to 1) = 0$.

(a) Generate 20 trajectories of this Markov process, all starting from $X_0 = 10$, up until the (random) time τ at which the absorbing state $X_\tau = 0$ is reached. Verify that once reaching that state, the process remains there.

(b) Estimate numerically the probability that the process reaches the absorbing state in the time interval $t \in [0, 10]$ starting from $X_0 = 5$. Thus the probability that you are trying to estimate is

$$P(\{X_t = 0 : t \in [0, 10]\}|X_0 = 5). \tag{2}$$

(c) Estimate numerically the expected time to absorption (expectation of the random time to reach 0) from the initial population $X_0 = 10$. That expectation mathematically is

$$T_{10}^{abs} = E[\tau_{10}], \tag{3}$$

where τ_{10} is the random absorption time from $X_0 = 10$.

(d) Repeat Part (c) for different starting points $X_0 = i$ and plot T_i^{abs} versus i for a suitable range of values. Analyse your results and explain them mathematically using the result of Q4(b).

Prize question

R100 for the best complete answer. Hand in your solution on a separate sheet.

This one is for the bio-maths guys to work on during recess, although everyone is welcome to try it. Simulate the predator-prey model presented in Jacobs Sec. 8.8 (the book's pdf is on SunLearn) and verify that the expected predator and prey populations satisfy the deterministic Lotka-Volterra equations. Derive these equations from the stochastic model following Q4 and simulate many trajectories so you can see fluctuations about the deterministic, expected behavior. What is the meaning of the Lotka-Volterra equations with respect to the stochastic predator-prey model? Which of the two models (stochastic or deterministic) are we expecting to observe in Nature?