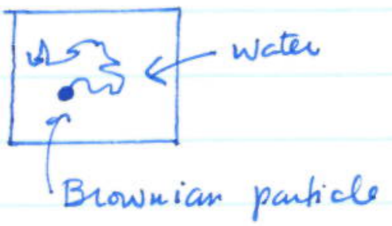
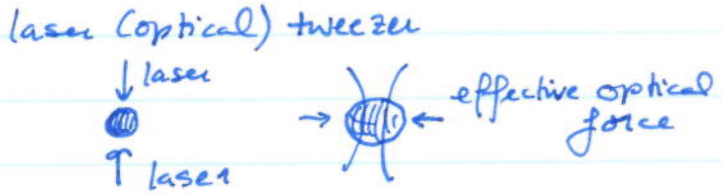
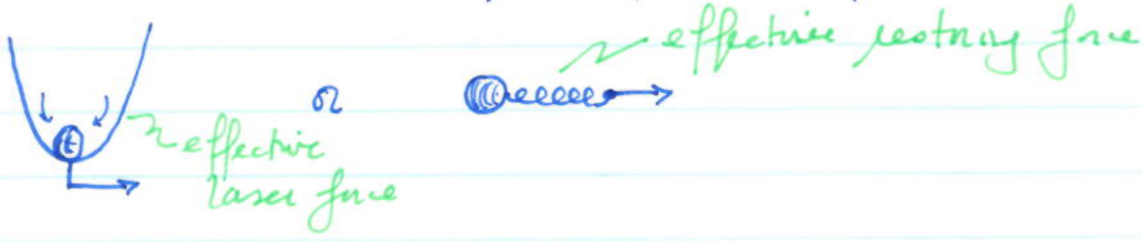
8.1. Dragged Brownian particle

Ref: R. van Zon, E.G.D. Cohen, PRE 67, 046102, 2003 for model.





- Low laser power: tweezer force is linear ⇒
- Effective model: Particle in a quadratic/parabolic potential



- Model: Noisy Newton's equation:

$$\dot{x}_t = v_t$$

$$m\dot{v}_t = \underbrace{-\alpha v_t}_{\text{Stoke friction}} - \underbrace{k\,(x_t - x_t^*)}_{\text{spring force}} + \underbrace{\xi_t}_{\text{noise}}$$

center of potential

- Noise model:

$$\langle \xi_t \rangle = 0$$

$$\langle \xi_t \xi_{t'} \rangle = \underbrace{2k_B T \alpha}_{\text{fluctuation-dissipation}} \; \delta(t-t')$$

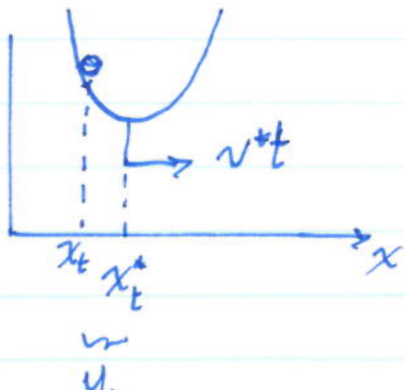
- Overdamped limit: $mk \ll \alpha^2$

$$\dot{x}_t = -\gamma\left(x_t - x_t^*\right) + \alpha^{-1}\xi_t \qquad \gamma = \frac{k}{\alpha} = \tau_r^{-1}$$

relaxation time

- Particular case: $x_t^* = v^* t$

drag velocity for potential



Potential frame: $y_t = x_t - x_t^*$
$ = x_t - v^* t$

- SDE:

$$\dot{y}_t = -\gamma y_t - v^* + \sigma \xi_t$$

i.e.

$$dY_t = \underbrace{-\gamma Y_t\, dt}_{\text{restoring force}} \underbrace{- v^* dt}_{\text{pull}} \underbrace{+ \sigma\, dW_t}_{\text{noise}}$$

- Linear SDE (Langevin or O-U) with drift.

- Observable:

$$W_T = \frac{1}{T}\int_0^T \underbrace{F}_{\text{spring force}} \overbrace{v^*}^{\text{pulling velocity}} dt$$

$$= \frac{1}{T}\int_0^T F \cdot dX^* = \frac{\text{work done by moving potential}}{T}$$

$$= \frac{1}{T}\text{work done by tweezer on BM particle}$$

i.e.

$$W_T = \frac{1}{T}\int_0^T v^*(-(X_t - X_t^*))\, dt$$

$$= -\frac{v^*}{T}\int_0^T y_t\, dt$$

$$= \frac{1}{T}\int_0^T f(Y_t)\, dt \qquad f(y) = -v^* y$$

linear additive process

- Large deviation calculation:

① Generator:

$$L = (-\gamma y - v^*)\frac{d}{dy} + \frac{\sigma^2}{2}\frac{d^2}{dy^2}$$

② Tilted generator:

$$L_k = L + kf$$

$$= (-\gamma y - v^*)\frac{d}{dy} + \frac{\sigma^2}{2}\frac{d^2}{dy^2} - v^* k y$$

non hermitian

③ SCGF = dom eigenvalue

$$L_k\, e(y) = \lambda(k)\, e(y)$$

Ref: Example 6.9 in HT2009

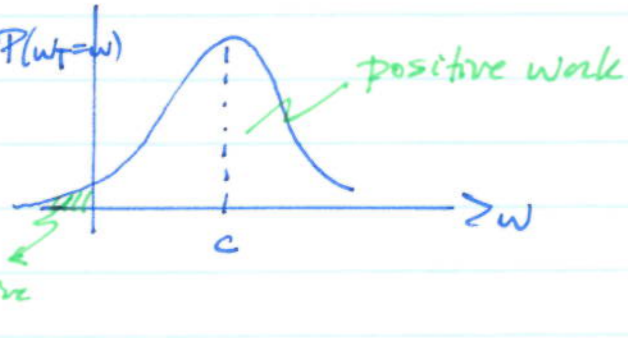
- Solution: $\gamma = 1$, $\sigma = 1$

$$\lambda(k) = ck + ck^2 = ck(1+k) \qquad c = v^{+2}$$

- GE Theorem:

$$P(W_T = w) \asymp e^{-T\,I(w)}$$

$$I(w) = \frac{(w-c)^2}{4c}$$



- Gaussian fluctuations!

$$\Pr(W_T > 0) > \Pr(W_T < 0)$$

asymmetry

- Fluctuation (a)symmetry (or relation)

$$\frac{P(W_T = w)}{P(W_T = -w)} \asymp \frac{e^{-T I(w)}}{e^{-T I(-w)}} = e^{T[I(-w) - I(w)]} = e^{w}$$

i.e. $$\lim_{T\to\infty} \frac{1}{T} \ln \frac{P(W_T = w)}{P(W_T = -w)} = \underbrace{I(-w) - I(w)}_{\substack{\text{rate function symmetry} \\ = \text{fluctuation relation}}} = w$$

- Same as

$$\lambda(k) = \lambda(-1-k)$$

cf HT2009 for refs on fluctuation relations

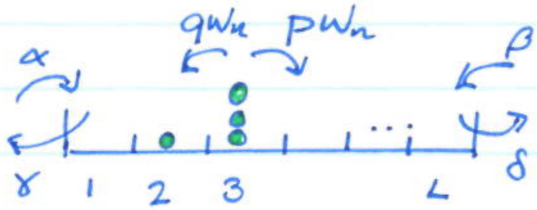
- Other observable: heat

$$Q_T = W_T - \Delta U_T = \underbrace{W_T}_{\text{work}} - \underbrace{(U_T - \underbrace{U_0}_{=0})}_{\Delta\text{ potential energy}}$$

$$U_T = \frac{k(x_t - x_t^*)^2}{2} \qquad \text{potential energy}$$

→ Exercise: LD for Q_T

8.2. Zero-range process



- L sites : $i = 1, 2, \ldots, L$
- n_i : occupation (particle number) at site i
- α, β : injection rates at boundaries (reservoirs)
- γ, δ : exit " " " "
- w_n : hopping rate : depends on site occupation n
- q : left asymmetry parameter
- p : right " "

- Example of rate : $w_n = 1 + \frac{b}{n^\alpha}$ $\quad$ $\alpha > 1 \to$ condensation particles pile up!

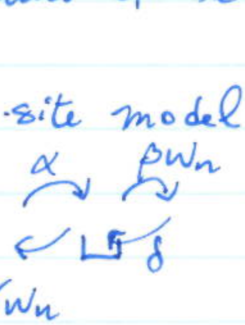
- Stationary current if $p \neq q$, $\alpha \neq \beta$, $\gamma \neq \delta$

→ Difficult to solve/analyze (beyond this course)

Refs :

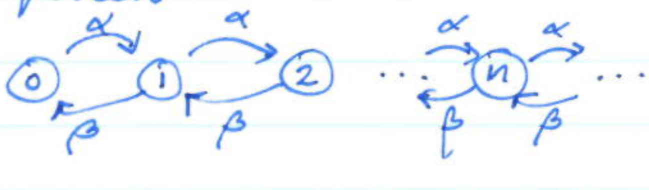
- Levine et al. J. Stat. Phys. 120, 759, 2005.
- Harris et al. J. Stat. Mech. P08003, 2005

- One-site model to do



Simple one site model

- Occupation : $n = 0, 1, 2, \dots$
- Probability for occupation: $p(n) = p_n$
- Probability vector: $\vec{p} = \begin{pmatrix} p_0 \\ p_1 \\ \vdots \end{pmatrix}$
- Markov process:



- Transition rates:

$$G = \begin{array}{cc} & i \\ j & \begin{pmatrix} -\alpha & \beta & 0 & \cdots \\ \alpha & -\alpha-\beta & \beta & \\ 0 & \alpha & -\alpha-\beta & \ddots \\ \vdots & 0 & \alpha & \\ & \vdots & 0 & \end{pmatrix} \end{array} = \begin{pmatrix} \ddots & \beta & 0 \\ \alpha & \ddots & \ddots \\ 0 & \ddots & \ddots \end{pmatrix}$$

normalization

- Evolution: $\vec{p}(t) = e^{tG}\vec{p}(0)$ or $\partial_t \vec{p}(t) = G\vec{p}(t)$
- Stationary state : $0 = G\vec{p}_s$ → solve coupled linear equations

- Current:
  - State in time : n_t
  - Injection of 1 particle : $n_{t^-} \to n_{t^+} = n_{t^-} + 1$ +1 current
  - Exiting " " " : $n_{t^-} \to n_{t^+} = n_{t^-} - 1$ −1 current

$$Q_T = \frac{1}{T} \sum_{t:\,\Delta n_t \neq 0} g(n_{t^-}, n_{t^+})$$

$$g(n, n') = \begin{cases} 1 & n' - n = 1 \\ -1 & n - n' = 1 \\ 0 & n = n' \\ 0 & |n - n'| > 1 \end{cases}$$

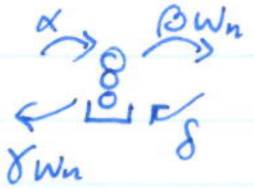
· Tilted generator

$$G_k = G\, e^{kg}$$

$$G_k = \begin{pmatrix} -\alpha & \beta e^{-k} & 0 & \cdots \\ \alpha e^{k} & -\alpha-\beta & \beta e^{-k} & \\ 0 & \alpha e^{k} & -\alpha-\beta & \ddots \\ \vdots & 0 & \alpha e^{k} & \ddots \end{pmatrix} = \begin{pmatrix} \diagdown & \beta e^{-k} & 0 \\ \alpha e^{k} & \diagdown & \diagdown \\ 0 & \diagdown & \diagdown \end{pmatrix}$$

→ Exercise:
- Obtain $\lambda(k)$
- " $I(q)$

One-side zero-range process



$n = 0, 1, 2, \ldots$

constant rates for leaving

→ Exercise: Redo calculations for this model with $w_n = 1$
then $w_n = n$

more chance to leave for higher occupation

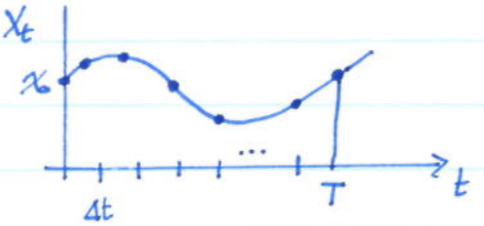
cf Harris 2005

8.3. Paths large deviations

Ref: HT2009, Sec. 6.1

- SDE: $dX_t = \underbrace{f(X_t)}_{\text{force, drift}}\,dt + \underbrace{\sqrt{\varepsilon}\,dW_t}_{\text{noise, diffusion}}$ Itô $X_0 = x_0$ initial cond.

- Sample path: $\{X_t\}_{t=0}^{T}$
- Discretization:



$\{X_t\}_{t=0}^{T}$ Markov process

$\downarrow$

$\{X_i\}_{i=0}^{n}$ $n = \frac{T}{\Delta t}$ Markov chain

- Joint path distribution:

$$p(\{X_i\}) = p(X_0, X_1, \ldots, X_n)$$

Markov chain

$$= p(X_0)\,\underbrace{\Pi_{\Delta t}(X_1|X_0)}_{\text{infinitesimal propagator of Week 6}}\,\Pi_{\Delta t}(X_2|X_1)\cdots\Pi_{\Delta t}(X_n|X_{n-1})$$

- Path distribution:

$$p[x] = p(\{X_t\}_{t=0}^{T}) = \lim_{n\to\infty} p(\{X_i\})$$

only a formal expression

- LDP in small noise limit:

$$p_\varepsilon[x] \asymp e^{-I[x]/\varepsilon} \qquad \varepsilon \to 0$$

path LDP

$$\lim_{\varepsilon\to 0} -\varepsilon \ln p_\varepsilon[x] = I[x]$$

- Rate function:

$$I[x] = \frac{1}{2}\int_0^T (\dot{x}_t - f(x_t))^2\,dt$$

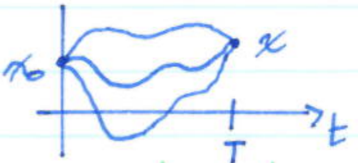
a functional

Also called action, entropy.

→ Exercise: Derive expressions of $p_\varepsilon[x]$ and $I[x]$ from SDE and $\Pi_{\Delta t}(x'|x)$.

· Transition probability

$$p(x,T|x_0,0) = \int_{x_0=x_0}^{x_T=x} D[x]\; P_\varepsilon[x]$$



↳ sum over paths = path integral

$$\asymp \int D[x]\; e^{-I[x]/\varepsilon}$$

LDP

$$\asymp e^{-\min_{x_t: x_0, x_T=x} I[x]/\varepsilon}$$

Laplace approximation

⇒ LDP for $p(x,T|x_0,0)$:

$$p(x,T|x_0,0) \asymp e^{-V(x,T|x_0,0)/\varepsilon}$$

$$\lim_{\varepsilon\to 0} -\varepsilon \ln p(x,T|x_0,0) = V(x,T|x_0,0)$$

· Rate function:

$$V(x,T|x_0,0) = \min_{\substack{\{x_t\}_{t=0}^T:\, x_0=x_0\\ x_T=x}} I[x]$$

contraction principle of Week 4

$$= I[x^*]$$

· Minimizing path:

$$x^* : \{x_t^*\}_{t=0}^T \text{ s.t. } I[x] \text{ is minimal}$$

Also called the most probable path or <u>instanton</u>

↙ from field theory

· Euler-Lagrange equation:

$$\delta I[x] = 0 \Rightarrow \begin{cases} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}} - \dfrac{\partial L}{\partial x} = 0 \\ x_0 = x_0 \\ x_T = x \end{cases}$$

2nd order ODE with 2 boundary conditions

where

$$L(x,\dot{x}) = \frac{1}{2}\left(\dot{x} - f(x)\right)^2 \qquad I = \int_0^T L(x_t,\dot{x}_t)\,dt$$

↳ Lagrangian

Example: O-U process or Langevin equation

$$dX_t = -\gamma X_t\, dt + \sqrt{\varepsilon}\, dW_t$$

$$p_\varepsilon(x,t\,|\,x_0,0) \asymp e^{-V(x,t\,|\,x_0,0)/\varepsilon}$$

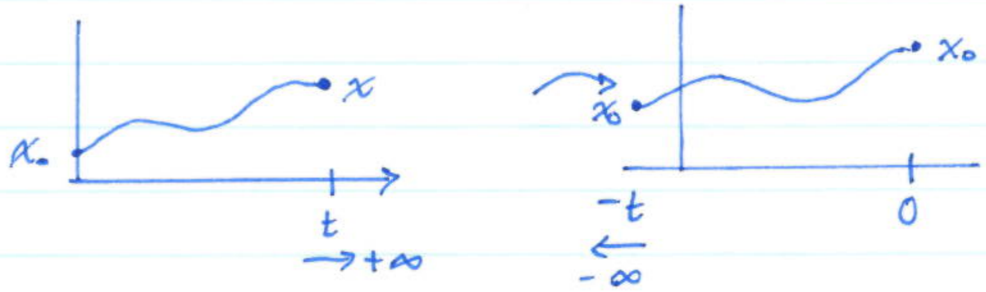
$$V(x,t\,|\,x_0,0) = \min_{x_0,\; x_t = x} \frac{1}{2}\int_0^t (\dot{x}_s + \gamma x)^2\, ds$$

→ Exercise: Find $V(x,t\,|\,x_0,0)$
" instanton.

Example: Stationary distribution of OU process

$$p_s(x) = \lim_{t\to\infty} p(x,t\,|\,x_0,0) \qquad \forall\, x_0 \quad \text{(ergodic limit)}$$

$$= \lim_{t\to\infty} p(x,0\,|\,x_0,-t)$$



$$p_s(x) \asymp e^{-V(x)/\varepsilon}$$

$$V(x) = \min_{\substack{x_{-\infty} = \text{whatever} \\ x_0 = x}} \frac{1}{2}\int_{-\infty}^0 (\dot{x}_s + \gamma x)^2\, ds$$

→ Exercise: Find $V(x)$
" instanton
Verify instanton is time reversed of $\dot{x} = -\gamma x$
i.e. it satisfies $\dot{x} = \gamma x$