The Donsker-Varadhan theory of large deviations

Hugo Touchette

School of Mathematical Sciences
Queen Mary, University of London

Grandes déviations et systèmes de particules en interaction, états stationnaires hors équilibre
Institut Henri Poincaré, Paris, France
6 Novembre 2012

Plan

1. Recap on large deviation theory
2. Different scaling limits
3. Donsker-Varadhan theory
4. Level-1, 2 and 2.5 large deviations
5. Mention 2 problems

The large deviation approach to statistical mechanics
Phys. Rep. 478, 1-69, 2009

Large deviation approach to nonequilibrium systems
HT, Rosemary J. Harris
arxiv:1110.5216

Large deviation theory

- Random variable: A_n
- Probability density: $P(A_n = a)$

Large deviation principle (LDP)

$$P(A_n = a) \approx e^{-nI(a)}$$

- Meaning of $\approx$:

$$\ln P(a) = -nI(a) + o(n)$$

$$\lim_{n\to\infty} -\frac{1}{n} \ln P(a) = I(a)$$

- Rate function: $I(a) \geq 0$

Goals of large deviation theory

1. Prove that a large deviation principle exists
2. Calculate the rate function

Obtaining LDPs

Gärtner-Ellis Theorem

- Scaled cumulant generating function:

$$\lambda(k) = \lim_{n\to\infty} \frac{1}{n} \ln E[e^{nkA_n}], \qquad k \in \mathbb{R}$$

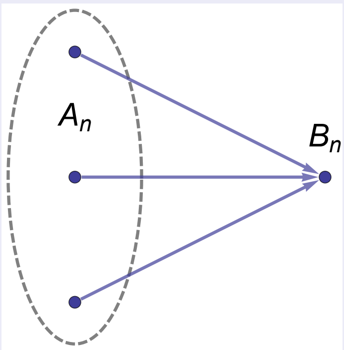
- If $\lambda(k)$ is differentiable, then

$$I(a) = \max_k \{ka - \lambda(k)\} = \text{Legendre transform of } \lambda(k)$$

Contraction principle

- $B_n = f(A_n)$
- LDP for $A_n \Rightarrow$ LDP for B_n:

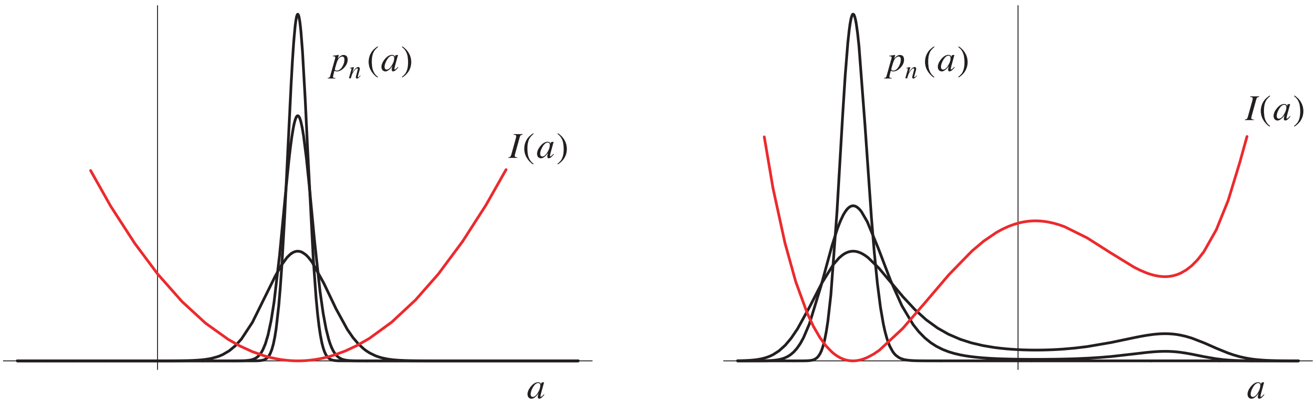
$$I_B(b) = \min_{a:f(a)=b} I_A(a) = \min_{f^{-1}(b)} I_A(a)$$



General properties

$$P(A_n = a) \approx e^{-nI(a)}$$

- Most probable value = typical value = min and zero of I
- Zero of I = Law of Large Numbers
- Local parabolic minimum = Central Limit Theorem



- LDT = Theory of typical states and fluctuations
- Requires scaling limit

Different scaling limits

Long-time limit (Donsker-Varadhan)

$$A_T = \frac{1}{T}\int_0^T f(X_t)\, dt, \qquad P(A_T = a) \approx e^{-T I(a)}$$

Low-noise limit (Freidlin-Wentzell)

$$dX_t = f(X_t)dt + \sqrt{\epsilon}\, dW_t, \qquad P[x] \approx e^{-I[x]/\epsilon}$$

Macroscopic (hydrodynamic) limit

- N particles evolving in volume L^d
- $N \to \infty$, $L \to \infty$, $\rho = N/L^d$ = const
- $x \to x/L$, $t \to t/L^2$ (diffusive scaling)

$$P[\rho(x,t)] \approx e^{-L^d I[\rho]}$$

- Many particles with Langevin dynamics [Gärtner-Dawson 1980s]
- Mixed limits

Long-time or steady-state LDPs

- Markov process: X_t
- Generator: L
- Additive observable (level-1):

$$A_T = \frac{1}{T}\int_0^T f(X_t)\, dt$$

- Current-like observable:

$$A_T = \frac{1}{T}\sum_{0\leq t\leq T:\Delta X_t\neq 0} g(X_{t^-}, X_{t^+})$$

- Mixed observable:

$$A_T = \frac{1}{T}\sum_{0\leq t\leq T:\Delta X_t\neq 0} g(X_{t^-}, X_{t^+}) + \frac{1}{T}\int_0^T f(X_t)\, dt$$

Long-time LDPs (cont'd)

$$P(A_T = a) \approx e^{-TI(a)}$$

Gärtner-Ellis

- SCGF:

$$\lambda(k) = \lim_{T\to\infty} \frac{1}{T} \ln E[e^{TkA_T}]$$

- GE Theorem:

$$I(a) = \max_k \{ka - \lambda(k)\}$$

Donsker-Varadhan

- Tilted operator:

$$L_k = L\, e^{kg} + kf$$

- Dominant eigenvalue: $\zeta(L_k)$
- SCGF: $\lambda(k) = \zeta(L_k)$

Example: Langevin equation

$$dX_t = -aX_t\,dt + \sigma\,dW_t$$

Linear observable

$$S_T = \frac{1}{T}\int_0^T X_t\,dt$$

- Tilted generator:

$$L_k = -ax\frac{d}{dx} + \frac{\sigma^2}{2}\frac{d^2}{dx^2} + kx$$

- Rate function:

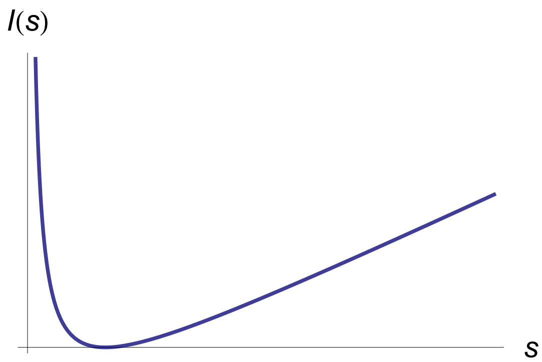
$$I(s) = \frac{a^2 s^2}{2\sigma^2}$$

Quadratic observable

$$S_T = \frac{1}{T}\int_0^T X_t^2\,dt$$

- Rate function:

$$I(s) = \frac{a^2 s}{2\sigma^2} - \frac{a}{2} + \frac{\sigma^2}{8s}$$

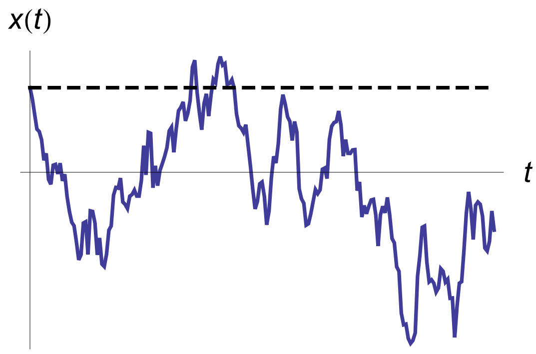


Empirical distribution (level-2 LDP)

Donsker & Varadhan 1960s

- Markov process: $\{X_t\}_{t=0}^T$
- Markov generator: L
- Empirical density:

$$\rho_T(x) = \frac{1}{T}\int_0^T \delta(X_t - x)\,dt$$



LDP

$$P(\rho_T = \rho) \approx e^{-TI(\rho)}$$

- Rate function:

$$I(\rho) = -\inf_{u>0} E_\rho\left[\frac{Lu}{u}\right] = -\inf_{u>0}\int dx\,\rho(x)\frac{(Lu)(x)}{u(x)}$$

- Equivalent to GE

Level-2 LDP: Remarks

- Other representation [Maes]:

$$I(\rho) = \sup_h \sum_{x,y} \rho(x)[W(x,y) - W_h(x,y)]$$

  - Tilted rates: $W_h(x,y) = e^{h(y)/2} W(x,y) e^{-h(x)/2}$

- Level-2 to level-1 contraction:

$$A_T = \frac{1}{T}\int_0^T f(X_t)\, dt = \int f(x)\, \rho_T(x)\, dx = a(\rho_T)$$

- Reversible systems (detailed balance):

$$I(\rho) = -\left\langle \sqrt{\frac{\rho}{p}}, L\sqrt{\frac{\rho}{p}} \right\rangle_p, \qquad p = \text{stationary dist.}$$

- Rate functions not explicit in general (minimization involved)

Current-density LDP (level-2.5 LDP)

Maes & Netočný 2007, 2008

- SDE:

$$dX_t = F(X_t)dt + \sigma dW_t$$

- Empirical current:

$$j_T(x) = \frac{1}{T}\int_0^T \delta(X_t - x) \circ dX_t = \frac{1}{T}\int_0^T \delta(X_t - x)\, \dot{X}_t\, dt$$

- Expected current:

$$E[j_T(x)] = F\rho_s(x) - \frac{\sigma^2}{2}\nabla\rho_s(x) = \text{Fokker-Planck current}$$

- Typical current:

$$j_T(x) \to j_s(x) = \text{FP current}$$

- What about fluctuations?

Current-density LDP (cont'd)

Joint LDP

$$P(\rho_T = \rho, j_T = j) \approx e^{-TI(\rho,j)}$$

- Rate function:

$$I(\rho,j) = \begin{cases} \frac{1}{2}\int (j - j_\rho)(\rho\sigma^2)^{-1}(j - j_\rho)(x)\, dx & \nabla\cdot j = 0 \\ \infty & \nabla\cdot j \neq 0 \end{cases}$$

- Fluctuating FP current: $j_\rho = F\rho - \frac{\sigma^2}{2}\nabla\rho$

- Current fluctuations are sourceless (in LD limit)
- Typical value:

$$j_s(x) = F\rho_s - \frac{\sigma^2}{2}\nabla\rho_s(x)$$

- Contractions:

$$I(\rho) = \min_j I(\rho,j), \qquad I(j) = \min_\rho I(\rho,j)$$

Conclusion, open problems

LDT = Complete theory of typical states and fluctuations

- Long-time (Donsker-Varadhan):
  - Largest eigenvalue problem
  - Rate functions not explicit in general - involves minimization
- Low-noise (Freidlin-Wentzell):
  - Min action path (instanton) problem
  - Saddle-point approximations of path integrals

Two problems

1. Sufficient / minimal observables
   - Observable with explicit rate function
   - Observable that completely characterizes a stochastic process
2. F-W vs D-V
   - $T \to \infty$, $\epsilon \to 0$ limits do not commute
   - T and ϵ trade-offs? [Paniconi, Oono PRE 1997]
   - When does F-W = D-V? [Speck, Engel, Seifert arxiv:1210.3042]