Legendre-Fenchel transform of convex and nonconvex functions

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Legendre and Fenchel

Adrien-Marie Legendre
1752-1833

Werner Fenchel
1905-1988

No
(that’s Louis)

Apparently, yes

\[ f^*(k) = kx_k - f(x_k) \]
\[ x_k : f'(x) = k \]

\[ f^*(k) = \sup \{ kx - f(x) \} \]

Can. J. Math. 1(1) 73, 1949
Two applications

Large deviation theory

\[ S_n = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad X_i \sim p(x), \text{ iid} \]

\[ p(S_n = s) \asymp e^{-nI(s)}, \quad I(s) = \sup_k \{ ks - \ln \langle e^{kX} \rangle \}. \]

Classical mechanics

Lagrangian mechanics

\[ L(x, \dot{x}) \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \]

Hamiltonian mechanics

\[ H(x, p) = p\dot{x} - L(x, \dot{x}) \]

\[ p = \frac{\partial L}{\partial \dot{x}} \]

\[ \dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x} \]
Plan

- Convex sets
- Lower semi-continuity
- Subdifferentials
- Convex functions
- Legendre-Fenchel transforms
- Duality properties
- Generalizations

Some notes (click on the links):

- Elements of convex analysis (HT)
- Legendre-Fenchel transforms in a nutshell (HT)
- A Course in Convex Analysis (A. Bossavit)