Doing maths with computers

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Presentation available at
http://www.maths.qmul.ac.uk/~ht

Outline

1 A brief history of computers
2 Crunching big numbers
3 Solving equations
4 Visualising functions
5 Experimental mathematics
6 Programming
Early mechanical calculators

- Abacus
- Mesopotamia (2500 BC), Roman Empire, China, etc.
- Quick arithmetic operations
- Pascal’s calculator 1645 (Pascalina or the Arithmetique)
- Blaise Pascal (1623-1662)
- Add and subtract numbers
- Difference engine 1822
- Charles Babbage (1791-1871)
- Compute values of polynomial functions

More mechanical computers

- Leonardo Torres Quevedo (1852-1936)
- Computes real and complex roots of trinomial equations $x^p + x + r = 0$
- Isograph, AT&T, 1937
- Calculates roots of polynomials up to degree 15
- Differential analyzer, 1910s-1920s
- Solves differential equations
First universal computers

1945: ACE (Automatic Computing Machine)
Designed by Alan Turing

1946: ENIAC (Electronic Numerical Integrator And Computer)
Designed by John von Neumann

Modern computers

Personal computers / calculators

Supercomputers

Fastest computer

- DOE Roadrunner, USA, 1400 TeraFlops = 1400 \times 10^{12} \text{ ops/sec}
- List at www.top500.org
Mathematical softwares

Maple (commercial)
Maplesoft (Canada)
maplesoft.com

Mathematica (commercial)
Wolfram Research (USA)
wolfram.com

Sage (open source, free)
sagemath.org

Magma (non-commercial, license fee)
University of Sydney
magma.maths.usyd.edu.au/magma/

- Numerical computations
- Symbolic computations

Representing large numbers

**Question 1**
What's the biggest number you can write on a typical pocket calculator?

**Solution:**

**Question 2**
What's the biggest number that you think a computer can represent?

**Solution:**

**Question 3**
What limits us in representing numbers on computers?

**Solution:**
Crunching big numbers

Question 4
Write down the value of 4!

Solution:

Question 5
How many digits do you think 1000! has?

Solution:

Question 6
Write the largest number you can think of using 4 digits arranged in any way you like.

Solution:

Solving simple polynomial equations

- Some equations can be solved by hand
- No need to use computers

Question 7
Write down the solution of $2x + 4 = 8$.

Solution:

Question 8
Write down the solutions of the general equation

$$ax^2 + bx + c = 0$$

where $a$, $b$, and $c$ are constants. How many solutions are there?

Solution:
Cubic equations

- Some equations are more difficult to solve
- Computers begin to be useful

**Question 9**
Find all the solutions of \( x^3 - x = 0 \).

**Solution:**

**Question 10**
Consider the general equation

\[
ax^3 + bx^2 + cx + d = 0
\]

where \( a, b, c, \) and \( d \) are constants. Do you know the formula for the solutions of this equation? How many solutions are there?

**From cubic to quintic equations**
Here’s one solution of the equation of degree 3 of the previous page:

\[
x = \frac{-\frac{b}{3a} + \frac{2^{1/3} \left(-b^2 + 3ac\right)}{3a^{2/3}}} {2^{1/3} \left(-b^2 + 3ac\right)} + \frac{\left(-2b^3 + 9abc - 27a^2d\right) + \left(4\left(-b^2 + 3ac\right)^3 + (-2b^3 + 9abc - 27a^2d)\right)}{2^{1/3} \left(-b^2 + 3ac\right)}
\]

**Question 11**
Consider the general equation

\[
a x^4 + b x^3 + c x^2 + d x + e = 0
\]
of degree 4. Is there an explicit formula for the solutions of this equation?

**Solution:**

**Mathematical fact**

Solutions of equations of degree 5 (quintics) cannot be expressed in terms of the four arithmetic operations and roots only.
Transcendental equations

- Some equations can’t be solved in closed form
- But they can be solved numerically on a computer

**Question 12**
Consider the equation

\[ e^{-x} = x. \]

Can you solve it? That is, can you find the numerical value that verifies this equation?

**Solution:**

**Question 13**
Find the solution of

\[ \cos(x) = x \]

in the interval \([0, \pi]\).

**Solution:**

**Question 14**
Find the two solutions of the equation

\[ \cos(x^3) - x^2 = 0. \]

**Solution:**
Plotting functions of one variable

Question 15

Can you plot the following functions?

a) \( f(x) = x^2 \)

b) \( f(x) = \sin(x) \)

c) \( f(x) = \sin\left(\frac{1}{x}\right) \)

Solution:

Plotting functions of two variables

Question 16

Try to plot the following functions:

a) \( f(x, y) = x^2 + y^2 \)

b) \( f(x, y) = x^2 - y^2 \)

c) \( f(x, y) = x \sin(x) \cos(y) \)

Solution:
Plotting function of three variables

- Can we plot a function \( f(x, y, z) \) of three variables?

**Question 17**

The equation of a sphere of radius \( r \) is

\[
x^2 + y^2 + z^2 = r^2
\]

Try to plot the sphere of radius 2 in 3D.

**Solution:**

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The prime numbers

**Question 18**

What's a prime number?

**Solution:**

---

**Question 19**

Write down the first 10 primes.

**Solution:**

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**Question 20**

Is 7918 a prime? What about 7919?

**Solution:**
Distribution of primes

Here are the primes between 2 and 100:

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**Question 21**
Do you see any order in the way the primes appear?

**Solution:**

**Question 22**
Is there a maximum prime number? In other words, is there a finite or an infinite number of prime numbers?

**Solution:**

**Plotting the distribution of primes**

- Cumulative distribution:

\[ N(x) = \text{Number of primes smaller or equal to } x \]

**Question 23**
Do you see any pattern in the way \( N(x) \) grows?

**Solution:**

**Question 24**
How quickly does \( N(x) \) grow?

**Solution:**
Order in the primes: The Ulam prime spiral

Stanislaw Ulam, 1963

Sacks’s spiral (1994)

Archimedean spiral

$x^2 + x + 41$ (Euler, 1772)
What’s programming?

- Instructing a computer to perform a task
- Writing down the recipe for a computation
- Computer’s language = **programming language**

**Programming language**

Java, C, C++, Pascal, Fortran, Basic,...

**Basis of all languages**

- Arithmetic operations: +, −, ×, ÷
- Functions: e.g. sin(x)
- Repeated execution (loops)
- Conditional execution (ifs)

A simple example

**Problem**

Add all the prime numbers between 1 and 100

**Recipe**

\[\text{sumvalue} = 0\]

1 prime? no
2 prime? yes \(\Rightarrow \text{sumvalue} + 2\)
3 prime? yes \(\Rightarrow \text{sumvalue} + 3\)
4 prime? no
...
99 prime? no
100 prime? no
Print final \text{sumvalue}

**Program**

\[
\text{sumvalue}:=0; \\
\text{for} \ i \ \text{from} \ 1 \ \text{to} \ 100 \ \text{do} \\
\quad \text{if isprime}(i)=\text{true} \ \text{then} \\
\quad \quad \text{sumvalue}:=\text{sumvalue}+i; \\
\quad \text{end if;} \\
\text{end do;} \\
\text{sumvalue;}
\]
Another example: The Lorenz attractor
Edward Lorenz, 1963

- Position: \((x(t), y(t), z(t))\)
- Equations of motion:
  \[
  \begin{align*}
  \frac{dx}{dt} &= \sigma(y - x) \\
  \frac{dy}{dt} &= x(\rho - z) - y \\
  \frac{dz}{dt} &= xy - \beta z
  \end{align*}
  \]

Basic code

for i from 0 to t/dt do
  x:=x+sigma*(y-x)*dt;
  y:=y+(rho*x-x*z-y)*dt;
  z:=z+(x*y-beta*z)*dt;
end do:

Extra: Program for cumulative distribution of primes

- Cumulative distribution:
  \[N(x) = \text{Number of primes smaller or equal to } x\]

Recipe

1. Scan all number from 1 to \(x\)
2. Add 1 to a counter when a prime is encountered

Algorithm

\[
\begin{align*}
\text{nprimes} &= 0 \\
1 \text{ prime? no} \\
2 \text{ prime? yes }\Rightarrow \text{nprimes} + 1 \\
3 \text{ prime? yes }\Rightarrow \text{nprimes} + 1 \\
4 \text{ prime? no} \\
\vdots \\
x \text{ prime?}
\end{align*}
\]
Print final \text{nprimes}

Program

nprimes:=0;
for i from 1 to x do
  if isprime(i)=true then
    nprimes:=nprimes+1;
  end if;
end do;
nprimes;
If you want to know more...

- History of computers:
  - Wikipedia

- Top 500 computers in the world:
  - http://www.top500.org

- Solving quintic equations:
  - http://library.wolfram.com/examples/quintic/

- Prime numbers:
  - Wikipedia

- Prime spiral:
  - Wikipedia
  - http://www.numberspiral.com/
  - http://mathworld.wolfram.com/PrimeSpiral.html

- Lorenz attractor:
  - Wikipedia
  - http://mathworld.wolfram.com/LorenzAttractor.html

http://www.maths.qmul.ac.uk/~ht

Notes