Solution to Problem 191.8 of M500-191

The infinite exponential $x^{x^{x^{\cdots}}}$ is the infinite iterate of the map $f(y) = x^y$ starting at $y = x$, and converges, as such, to the *stable* fixed-point of f satisfying $y = x^y$. For $x = 1.1$, there are two fixed-points, namely, $y = 1.111782011\ldots$ and $y = 38.22873285\ldots$, as mentioned in the problem, but only the first one is stable; the second is unstable. Thus, denoting by $f^{(n)}(x)$ the n-fold composition of the map f starting at x, we must have

$$x^{x^{x^{\cdots}}} = \lim_{n\to\infty} f^{(n)}(x) = 1.111782011\ldots$$

for $x = 1.1$. The convergence is quite rapid, as can be checked by calculating the first few iterates:

$$\begin{aligned} 1.1^{1.1} &= 1.110534241\ldots \\ 1.1^{1.1^{1.1}} &= 1.111649800\ldots \\ 1.1^{1.1^{1.1^{1.1}}} &= 1.111768002\ldots. \end{aligned}$$

Incidentally, if we start off with any number for the initial value of the iterate, we end up with the same infinite exponential because $y = 1.111782011\ldots$ is the only stable fixed point of f. Thus

$$\lim_{n\to\infty} f^{(n)}(x_0) = 1.111782011\ldots$$

for any real x_0. In words this means that the infinite exponential is insensitive to the number put at the top (infinite) level of exponentiation.

Hugo Touchette
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