Chapter 5 : Stochastic differential equations

· Jacobs, Chaps 2, 3

· GS, Chap. 13

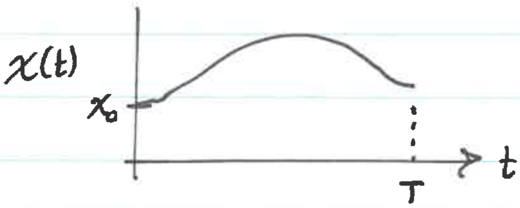
5.1 Introduction

Ordinary differential equations
ODEs

$$\begin{cases} \dfrac{dx(t)}{dt} = f(x(t), t) \\ x(0) = x_0 \end{cases}$$

$$\dot{x} = \underbrace{f(x,t)}_{\text{"force"}}$$

deterministic evolution



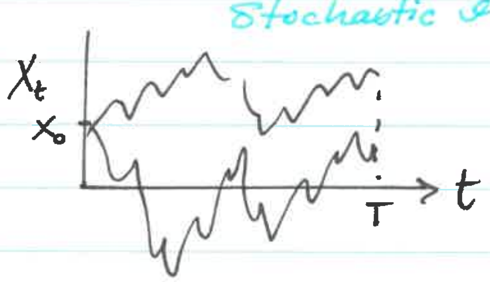
Stochastic differential equations
SDEs

random variable

$$\begin{cases} \dfrac{dX_t}{dt} = f(X_t, t) + \underbrace{\xi_t}_{\text{noise}} \\ X_0 = x_0 \end{cases}$$

$$\dot{X} = \underbrace{f(X,t)}_{\substack{\text{"force"} \\ \text{deterministic}}} + \underbrace{\xi}_{\text{noise}}$$

Stochastic evolution



5.2 Ordinary differential equations (ODEs)

· 1^{st} order ODE :

$$\dot{\vec{x}}(t) = \vec{f}(\vec{x}(t)) \quad , \quad \vec{x}(0) = \vec{x}_0$$

· State : $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ $\qquad \vec{x}(0) = \begin{pmatrix} (x_0)_1 \\ (x_0)_2 \\ \vdots \\ x(0)_n \end{pmatrix}$

· Force or drift : $\vec{f} = \begin{pmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \end{pmatrix} = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ f_2(x_1, \dots, x_n) \\ \vdots \end{pmatrix}$

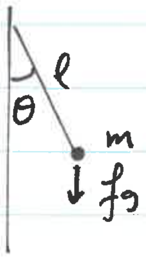
· Any n^{th} order ODE for $x(t) \in \mathbb{R}$ can be transformed to 1^{st} order ODE for $\vec{y}(t) \in \mathbb{R}^n$.

Example: Newton's equation $\quad F = ma = m\dot{v} = m\ddot{x}$

1st order $\quad$ 2nd order

$$\dot{x}(t) = v(t) \quad = f_x(x,v)$$
$$\dot{v}(t) = F/m \quad = f_v(x,v)$$

Example: Pendulum



$$m\ddot{\theta} + \frac{mg}{\ell}\sin\theta = 0$$

$$\ddot{\theta} + \omega^2 \sin\theta = 0 \qquad \omega = \sqrt{\frac{g}{\ell}} \text{ natural frequency}$$

$$\Rightarrow \quad \dot{\theta} = \varphi \qquad = f_\theta(\theta,\varphi)$$ Angular velocity
$$\dot{\varphi} = \ddot{\theta} = -\omega^2 \sin\theta \qquad = f_\varphi(\theta,\varphi)$$

$$\Rightarrow \quad \begin{pmatrix} \dot{\theta} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} \varphi \\ -\omega^2 \sin\theta \end{pmatrix} \qquad \begin{aligned} f_\theta(\theta,\varphi) &= \varphi \\ f_\varphi(\theta,\varphi) &= -\omega^2 \sin\theta \end{aligned}$$

5.3 Numerical solutions of ODEs

· Matlab: ode45 ~ Runge-Kutta

① Define f as function

order t, x important

```
function dxdt = f(t,x)
        dxdt = zeros(2,1);
        dxdt(1) = ... ;
        dxdt(2) = ... ;
end
```

or

```
function dxdt = f(t,x)
        dxdt = [ ...     ... ];
end
```

② Use f in solver

```
tspan = [0, 10];
x10 = ... ;
x20 = ... ;
[T, X] = ode45(@f, tspan,
               [x10 x20]);
plot(T, X(:,1), T,X(:,2))
```

time vector $\quad$ x, vector, traj in time

· Python : odeint from scipy.integrate

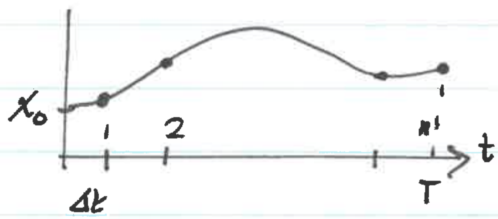
```
def f(x,t):        <- not same order as in Matlab
    :   parameters

    dx1dt = ...
    dx2dt = ...

    return [dx1dt, dx2dt]
```

```
tfinal = 10.0
dt = 0.01
tspan = np.linspace(
        0, tfinal, int(tfinal/dt))
x0 = [..., ...]
x = odeint(f, x0, tspan)
plt.plot(tspan, x[:,0])
```

· Euler scheme :



$$\begin{cases} \dfrac{dx(t)}{dt} = f(x(t)) \\ x(0) = x_0 \end{cases}$$

· Approximate $\{x(t)\}_{t=0}^{T}$ with n points $\{x_i\}_{i=0}^{n}$

· $n = T/dt$

$$\Delta x(t) = x(t+\Delta t) - x(t) = f(x(t))\,\Delta t \qquad \text{Difference eq.}$$

$$\Rightarrow \quad x(t+\Delta t) = x(t) + f(x(t))\,\Delta t + O(\Delta t^2)$$

$$\Rightarrow \quad \underset{\text{next}}{x_{i+1}} = \underset{\text{present}}{x_i} + f(x_i)\,\Delta t \qquad \text{Euler scheme}$$

· Code :

```
T = 10.0
dt = 0.01
n = int(T/dt)
x = zeros(n+1)
x[0] = ...
for i=0:n-1
    x[i+1] = x[i] + f(x[i])*dt
end
tspan = [0 : dt : T]
plot(tspan, x)
```

· Note : Total error

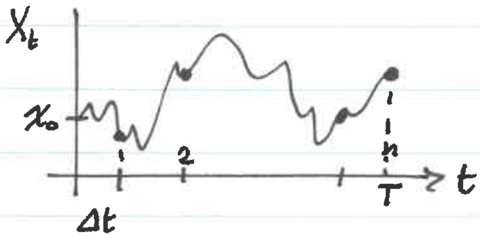
$$Err = \text{error per step} \times \text{nb steps} = \Delta t^2 \times \frac{T}{\Delta t} \sim \Delta t$$

· Note : Other methods :

- Mid point
- Implicit
- Verlet
- Runge-Kutta
- ...

5.4 Stochastic differential equations (SDEs)

$$\begin{cases} \dfrac{dX_t}{dt} = \underset{\text{det.}}{f(X_t)} + \underset{\text{noise}}{\xi_t} \\ X_0 = x_0 \end{cases}$$



· Discretization :

$$\Delta X_t = f(X_t)\,\Delta t + \xi_t \,\Delta t$$

$$= \underbrace{f(X_t)\,\Delta t}_{\text{Euler}} + \underbrace{\Delta W_t}_{\text{Increment of Wiener process/BM}}$$

↳ Gaussian white noise

$$\Rightarrow \quad X_{t+\Delta t} = X_t + f(X_t)\Delta t + \Delta W_t$$

$$\Delta W_t \sim \mathcal{N}(0,\Delta t) = \sqrt{\Delta t}\,\mathcal{N}(0,1)$$

$$\Rightarrow \quad \begin{cases} X_{i+1} = X_i + f(X_i)\Delta t + \sqrt{\Delta t}\, Z_i \\ Z_i \sim \mathcal{N}(0,1) \end{cases}$$

Euler–Maruyama scheme

· Note : Noise chosen to have variance ~ time. Other choices of noise possible : e.g., colored noise (correlated noises), Lévy noise, Poisson noise, etc.

· Ito notation for SDEs :

$$\dot{X}_t = f(X_t) + \xi_t$$
$$= f(X_t) + \frac{dW_t}{dt}$$

But W_t is nowhere differentiable! dW_t/dt doesn't make sense. Write instead as

$$dX_t = f(X_t)\,dt + dW_t$$

Same as $\Delta X_t = f(X_t)\,\Delta t + \Delta W_t$ Difference equation Well defined.

- Solution of SDE:

$$dX_t = f(X_t)dt + dW_t \Rightarrow X_t = X_0 + \int_0^t f(X_s)ds + \int_0^t dW_s$$

$$= X_0 + \int_0^t f(X_s)ds + W_t$$

implicit eq.

- Example: Add Gaussian white noise to linear eq.

$$\begin{cases} \dot{x}(t) = -\gamma x(t) \\ x(0) = x_0 \end{cases}$$

$$\hookrightarrow \quad \dot{x}(t) = -\gamma x(t) + \xi_t \qquad \text{Noisy ODE}$$

i.e. $dX_t = -\gamma X_t \, dt + dW_t$ SDE $\quad dW_t \sim \mathcal{N}(0, dt)$

```
tfinal = 10.0
dt = 0.01
n = int(tfinal/dt)
x = zeros(n+1)
x[0] = ...
for i in range(n):
    x[i+1] = x[i] + gamma*x[i]*dt + √dt *randn()
end
tspan = [0 : dt : tfinal]
plot(tspan, x)
```

- Exercise: Change dt, x_0, t_{final}, ...

- Example: Where to put noise in a Newton equation?

$$ma = m\ddot{x} = \underbrace{\text{Force} + \text{noise}}_{F_{tot}}$$

$$\Rightarrow \quad \dot{x} = v \qquad \leftarrow \text{no noise on velocity}$$

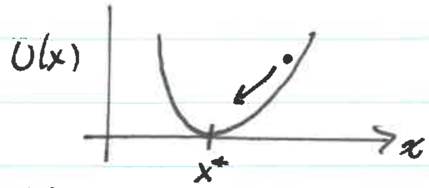
$$\dot{v} = \frac{F_{tot}}{m} = \frac{1}{m}(\text{Force} + \text{noise}) \qquad \leftarrow \text{noise in force/acceleration}$$

· Example : Noisy pendulum

$$d\theta_t = \varphi_t \, dt \qquad \leftarrow \text{no noise}$$

$$d\varphi_t = -\omega^2 \sin\theta_t \, dt + dW_t \qquad \leftarrow \text{Noise in acceleration}$$

· Example : Gradient dynamics



- $\vec{x}(t) \in \mathbb{R}^n$
- $\dot{\vec{x}}(t) = -\nabla U(\vec{x}(t))$
- Fixed point : $\dot{\vec{x}}^* = 0 \iff \vec{x}^*$ is a min of $U(\vec{x})$ (assuming not starting at other critical point)

· SDE version :

$$dX_t = -\nabla U(X_t)\,dt + \sigma\, dW_t$$

↳ noise amplitude

See CW5.

5.5 Fokker-Planck equation

· SDE :

$$\begin{cases} dX_t = F(X_t)\,dt + \sigma(X_t)\,dW_t \\ X_0 = x_0 \end{cases}$$

$X_t \in \mathbb{R}^n$

$F : \mathbb{R}^n \to \mathbb{R}^n$

$W_t \in \mathbb{R}^m$

σ $n \times m$

- Most cases: $m = n$ so σ is square matrix
- Consider here : σ is constant matrix

· Probability density of X_t : $p(x,t) = p(X_t = x)$

- X_t continuous space + continuous time
- X_t called a <u>diffusion</u>

- What is the evolution equation for $p(x,t)$?

· Evolution equation:

$$\begin{cases} \frac{\partial}{\partial t} p(x,t) = -\nabla\cdot(F(x)\,p(x,t)) + \frac{1}{2}\nabla\cdot D\nabla p(x,t) \\ p(x,0) = \delta(x_0) \end{cases}$$

$\nabla : D p$

or

$$\begin{cases} \frac{\partial}{\partial t} p(x,t) = -\sum_i \frac{\partial}{\partial x_i} F_i(x)\,p(x,t) + \frac{1}{2}\sum_{ij}\frac{\partial}{\partial x_i} D_{ij}\frac{\partial}{\partial x_j} p(x,t) \\ p(x,0) = \delta(x_0) \end{cases}$$

$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} D_{ij}\, p(x,t)$

- $D = \sigma\sigma^T$
- Called: Fokker-Planck or forward Kolmogorov equation.
- Similar to master equation: $\partial_t p = \mathcal{L} p$

linear operator

· Diagonal D: $D = \sigma^2 \mathbb{1}$

$$\Rightarrow \partial_t p = -\nabla\cdot F p + \frac{\sigma^2}{2}\Delta p$$

· 1D case: $\partial_t p = -\frac{\partial}{\partial x} F(x) p + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} p$

· Proof / derivation: Jacobs Sec 7.1 p. 103, Pavliotis Sec 2.5 p48.

· Example: BM $F = 0$, $\sigma = 1$

$$\partial_t p = \frac{1}{2}\Delta p$$

Heat equation

$$p_{BM}(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

· Example: Drifted BM $F = \mu$ $\sigma > 0$

$$\partial_t p = -\partial_x \mu p + \frac{\sigma^2}{2}\partial_{xx} p$$

$$= -\mu\,\partial_x p + \frac{\sigma^2}{2}\partial_{xx} p$$

$$p(x,t) = p_{BM}(x-\mu t, t)$$

· Example : Langevin equation

· Noisy ODE :
$$m \frac{d^2 x(t)}{dt^2} = \underbrace{F(x)}_{\text{force}} - \underbrace{\gamma \frac{dx(t)}{dt}}_{\substack{\text{friction} \\ -\gamma v}} + \underbrace{\sigma \cancel{\frac{d\xi(t)}{dt}}}_{\substack{\text{Gaussian white} \\ \text{noise}}}$$

· SDE :
$$dX_t = V_t \, dt$$
$$dV_t = \frac{F}{m} dt - \frac{\gamma}{m} V_t \, dt + \sigma_v \, dW_t$$

· State : (X_t, V_t)

· Distribution : $p(x,v,t) = p(X_t = x, V_t = v)$

· Fokker-Planck eq. :
$$\partial_t p(x,v,t) = -\partial_x F_x p(x,v,t) - \partial_v F_v p(x,v,t) + \frac{\sigma_x^2}{2} \partial_{xx} p(x,v,t) + \frac{\sigma_v^2}{2} \partial_{vv} p(x,v,t)$$

$$\Rightarrow \quad \partial_t p = \underbrace{-\partial_x v p - \partial_v \left(\frac{F}{m} - \frac{\gamma}{m} v\right) p}_{\text{advection terms}} + \underbrace{\frac{\sigma_v^2}{2} \partial_{vv} p}_{\substack{\text{diffusion only} \\ \text{on } v}}$$

· Example : Overdamped equation. Neglect $\ddot{x}$ term.

$\Rightarrow \quad 0 = F - \gamma \dot{x} + \sigma \xi_t$

$$\Rightarrow \quad dX_t = \frac{F}{\gamma} dt + \frac{\sigma}{\gamma} dW_t$$

· Force : $F = F(x)$ (other examples ...)

$ = -\nabla U$ (gradient)

· Distribution : $p(x,t) = p(X_t = x)$ — x only, no v

· F-P eq. :
$$\partial_t p = -\partial_x \frac{F}{\gamma} p + \frac{\sigma^2}{\gamma^2 2} \partial_{xx} p$$

5.6 Ergodic diffusions

· Stationary distribution: $p_s(x)$ such that $\partial_t p_s(x) = 0$

$$\Rightarrow \quad -\nabla\cdot F p_s + \frac{1}{2}\nabla\cdot \cancel{D\nabla} p_s = 0$$

$$\frac{1}{2} D \nabla^2 p_s$$

Diagonal case: $\quad -\nabla\cdot F p_s + \frac{\sigma^2}{2}\Delta p_s = 0$

1D case: $\quad -\partial_x F p_s + \frac{\sigma^2}{2}\partial_{xx} p_s = 0$

· Ergodic process:

$$\lim_{t\to\infty} p(x,t) = p_s(x)$$

for all initial distribution $p(x,0)$

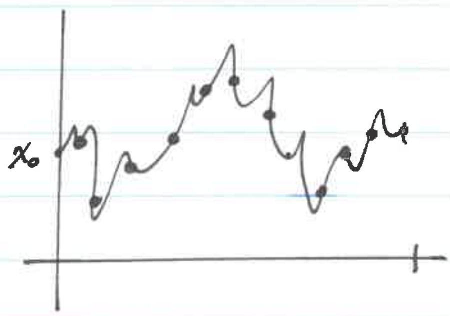
· Unique stationary distribution
· Reached in long-time limit

· Simulation:

1- Parallel simulations / trajectories
2- Long, sequential trajectory

see Chap 3

vs



5.7 Stochastic calculus

GS Secs 13.7, 13.8, Jacobs Secs 3.4, 3.5

- SDE:

$$dX_t = F(X_t)\,dt + \sigma(X_t)\,dW_t$$

Itô SDE
(infinitesimal difference eq.)

- Solution:

$$X_t = X_0 + \int_0^t F(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$$

random

  - Stochastic integral
  - Not "standard" Riemann integral
  - Requires different calculus rules

- Example: $\int_0^T dW_t = W_T - W_0 = W_T$

- Example: $x(t)$ smooth function of t. Then

$$\int_0^T x(t)\,dx(t) = \frac{1}{2}\int_0^T dx(t)^2 = \frac{1}{2}\, x(T)^2$$

Standard calculus

- Example: $\int_0^T W_t\,dW_t \neq \frac{1}{2} W_T^2$

In fact: $\int_0^T W_t\,dW_t = \frac{1}{2}\left(W_T^2 \underbrace{- T}_{\text{extra term}}\right)$

Stochastic calculus

Jacobs p. 33–

- Example: $\int_0^T (dW_t)^2$
  - Expectation: $E\left[\int_0^T (dW_t)^2\right] = \int_0^T E[(dW_t)^2] = \int_0^T dt = T$
  - Variance: $\mathrm{var}\left(\int_0^T (dW_t)^2\right) = \mathrm{var}\left(\sum_{i=0}^{n-1} (\Delta W)^2\right)$

$$= \sum_{i=0}^{n-1} \mathrm{var}(\Delta W)^2$$

$$= \sum_{i=0}^{n-1} 2\,\Delta t^2$$

Variance of Gauss² (χ^2 distribution)

$$= 2n\left(\frac{T}{n}\right)^2 = \frac{2T^2}{n} \searrow 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \int_0^T (dW_t)^2 = T = \int_0^T dt \quad \text{with probability 1}$$

$$\Rightarrow \quad dW_t^2 = dt$$

Itô's rule

- Itô's formula:
  - SDE: $dX_t = F(X_t)\,dt + \sigma(X_t)\,dW_t$
  - Transformation: $Z_t = f(X_t)$ — Assumed smooth + invertible $X_t = f^{-1}(Z_t)$
  - New SDE:

$$dZ_t = \Big(\underbrace{F f'}_{\text{normal calculus}} + \underbrace{\tfrac{1}{2}\sigma^2 f''}_{\text{extra term / Itô term}}\Big)dt + f'\sigma\, dW_t$$

$$dZ_t = \underbrace{G(Z_t)\,dt}_{\text{new drift}} + \mu(Z_t)\,dW_t$$

- Note: Deterministic "standard" calculus term:

$$\dot{x}(t) = F(x(t)) \qquad \Rightarrow \qquad \dot{z}(t) = f'(x(t))\,\dot{x}(t)$$

$$z(t) = f(x(t)) \qquad\qquad = f' F$$

- "Proof" (idea behind formula):

$$Z_t = f(X_t)$$

$$\Rightarrow dZ_t = f'(X_t)\,dX_t + \tfrac{1}{2} f''(X_t)\,dX_t^2 + \underbrace{\cdots}_{\text{higher order}}$$

$$= f'(F\,dt + \sigma\,dW) + \tfrac{1}{2} f''(F\,dt + \sigma\,dW)^2 + \cdots$$

$$= f' F\,dt + f'\sigma\,dW + \tfrac{1}{2} f'' F^2 \underbrace{dt^2}_{\text{higher order}} + \overbrace{\tfrac{1}{2} f''\sigma^2 \underbrace{dW^2}_{dt}}^{\text{Itô term}} + \underbrace{\tfrac{1}{2} f'' 2 F\sigma\, dt\,dW}_{\text{higher order}} + \cdots$$

$$= \left(f'F + \tfrac{1}{2} f''\sigma\right)dt + f'\sigma\,dW + \cdots$$

∎

- Note: $\sigma = 0 \Rightarrow dZ_t = F f'\,dt$ — Standard calculus

· Example: $Z_t = W_t^2$

1- SDE: $dW_t = dW_t$ $\qquad F=0, \quad \sigma=1$

2- Function: $Z_t = f(W_t)$ $\qquad f(w) = w^2 \qquad f' = 2w \qquad f'' = 2$

$$\Rightarrow \quad dZ_t = \left(F f' + \tfrac{1}{2}\sigma^2 f''\right)dt + f'\sigma\, dW_t$$

$$= \underbrace{dt}_{\text{Itô term}} + \underbrace{2W_t\, dW_t}_{\text{normal calculus}}$$

$$\Rightarrow \quad Z_T = \underbrace{Z_0}_{=0} + \int_0^T dt + 2\int_0^T W_t\, dW_t$$

$$= T + 2\int_0^T W_t\, dW_t$$

But $dZ_t = dW_t^2$ s.o. $Z_T = W_T^2$

$$\Rightarrow \quad \int_0^T W_t\, dW_t = \frac{W_T - T}{2}$$

· Example: $dY_t = \lambda Y_t\, dW_t$

$$\Rightarrow \quad Y_t = e^{\lambda W_t - \frac{\lambda^2}{2}t}$$

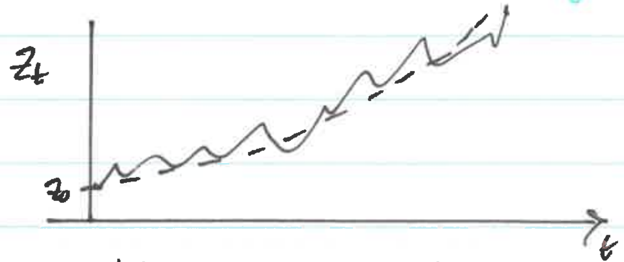
Standard calculus would give

$$\frac{dy_t}{y_t} = \lambda\, dw_t \quad \rightarrow \quad y_t = y_0 e^{\lambda w_t}$$

· Example: $dZ_t = \mu Z_t\, dt + \sigma Z_t\, dW_t$ $\qquad$ Geometric BM

Model of Stock prices

$$\Rightarrow \quad Z_t = Z_0\, e^{\left(\underbrace{\mu - \frac{\sigma^2}{2}}_{\text{corrected drift}}\right)t + \sigma W_t}$$



cf CW5

· Example: $dX_t = \mu dt + \sigma dW_t$

$Z_t = Z_0 e^{X_t}$ $\qquad$ To do. cf CW5

· Example: Ornstein-Uhlenbeck process

$$dX_t = -\gamma X_t\, dt + \sigma\, dW_t \qquad \gamma, \sigma > 0$$

Take

$$Y_t = X_t\, e^{\gamma t} = f(X_t, t)$$

↳ added time dependence

$$\Rightarrow \quad dY_t = \Big(\partial_t f + F \partial_x f + \frac{\sigma^2}{2} \underbrace{\partial_{xx} f}_{=0} \Big) dt + \partial_x f\, \sigma\, dW$$

$$= \Big(\underbrace{\gamma e^{\gamma t} X_t}_{Y_t} + \underbrace{(-\gamma X_t) e^{\gamma t}}_{Y_t} \Big) dt + e^{\gamma t} \sigma\, dW$$

$$= e^{\gamma t} \sigma\, dW_t$$

$$\Rightarrow \quad Y_t = Y_0 + \int_0^t e^{\gamma s} \sigma\, dW_s$$

$$\Rightarrow \quad X_t = e^{-\gamma t} Y_t$$

$$= e^{-\gamma t} Y_0 + \sigma e^{-\gamma t} \int_0^t e^{\gamma s}\, dW_s \qquad Y_0 = X_0 e^0 = X_0$$

$$= e^{-\gamma t} X_0 + \sigma \int_0^t e^{\gamma (s-t)}\, dW_s$$

· Note: More examples in Jacobs p.53.

· Note: $\int_0^T f(X_t)\, dX_t$ gives different results depending on discretization (Riemann) sum used: left-pt, right-pt, etc. In normal calculus, all discretizations give same result.

↳ Different calculus for integrals.

↳ Related to different rules for derivatives (Itô's formula)

See Higham's paper on sum learn.

5.8 Physical Brownian motion

Jacobs Sec 5.7

- Particle (tiny) moving in fluid
- a = diameter (cross section)
- η = fluid viscosity

- Friction force: $F_{friction} = -\gamma p = -\gamma m v$
- Stokes law: $\gamma = \frac{6\pi\eta a}{m}$

- Newton's equation: $m\ddot{x} = F_{friction} + F_{random}$

$$= -\gamma m \dot{x} + \sigma \underbrace{\xi(t)}_{\text{Gaussian white noise}}$$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} 0 & 1/m \\ 0 & -\gamma \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ \sigma\xi(t) \end{pmatrix}$$

or

$$\begin{pmatrix} dX_t \\ dP_t \end{pmatrix} = \begin{pmatrix} 0 & 1/m \\ 0 & -\gamma \end{pmatrix}\begin{pmatrix} X_t \\ P_t \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW_t$$

2D linear equation

- Equation for momentum: $dP_t = -\gamma P_t\, dt + \sigma\, dW_t$

Ornstein – Uhlenbeck process

- Solution: $P_t = e^{-\gamma t} P_0 + \sigma \int_0^t e^{-\gamma(t-s)} dW_s$
  - Verify with Itô's formula.
  - Expectation: $E[P_t] = e^{-\gamma t} E[P_0] + \sigma \int_0^t e^{-\gamma(t-s)} \underbrace{E[dW_s]}_{=0}$

$$= e^{-\gamma t} E[P_0] \xrightarrow{t\to\infty} 0$$

- Variance: $\mathrm{Var}(P_t) = \sigma^2 \int_0^t e^{-2\gamma(t-s)} dt$

$$= \frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma t}\right)$$

$$\xrightarrow{t\to\infty} \frac{\sigma^2}{2\gamma}$$

· Thermodynamic equipartition:

Boltzmann's constant

Temperature

$$\langle E \rangle = \frac{\langle p^2 \rangle}{2m} = \frac{k_B T}{2} \times \underbrace{2}_{\text{2 dimensions / degrees of freedom}}$$

$$\langle p^2 \rangle = \mathrm{var}(P_\infty)$$

$$\Rightarrow \quad k_B T = \frac{\mathrm{var}(P_\infty)}{2m} = \frac{\sigma^2}{4m\gamma}$$

$$\Rightarrow \quad \sigma = \sqrt{4 m \gamma k_B T} = \sqrt{24 \pi \eta a \, k_B T}$$

$$\Rightarrow \quad \sigma \propto \sqrt{T}$$

- Thermal noise
- Fluctuation–dissipation theorem
- Dissipation (friction) related to noise amplitude (fluctuation)

· Solution for X_t: $\quad X_t = \int_0^t \frac{P_s}{m}\, ds$

$$E[X_t] = \frac{1}{m\gamma}\left(1 - e^{-\gamma t}\right) E[P_0] \to \frac{E[P_0]}{m\gamma} = 0$$

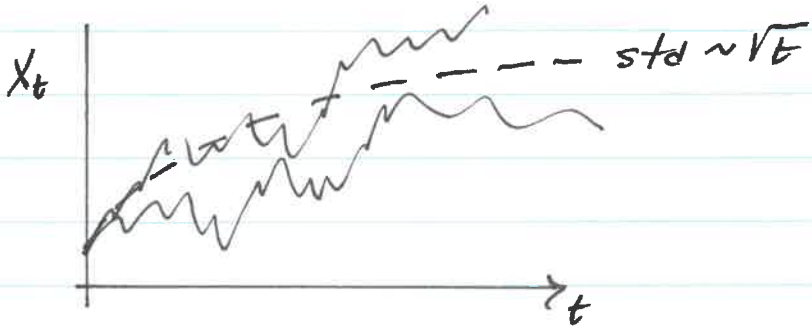
$$\mathrm{var}(X_t) = \frac{\sigma^2 t}{(m\gamma)^2} + \frac{\sigma^2}{2m^2\gamma^3}\left(4e^{-\gamma t} - e^{-2\gamma t} - 3\right)$$

$$\sim \frac{\sigma^2 t}{(m\gamma)^2}$$

$$= \left(\frac{2 k_B T}{3\pi \eta a}\right) t$$

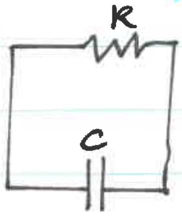
$$= D t$$

var ~ linear in time



5.9 Johnson-Nyquist noise

Johnson, Thermal agitation of electricity, 1928 (experiment)
Nyquist, 1928 (theory)
Van Kampen p. 222



$$I = \frac{dQ_t}{dt} \qquad C = \frac{Q_t}{V_t} \qquad V_t = R I_t$$

$$I = \frac{V}{R} = \frac{Q}{RC}$$

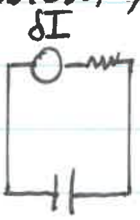
- Evolution equation: $\displaystyle \frac{dQ_t}{dt} = -\frac{Q_t}{RC} + \underbrace{\text{noise}}_{\delta I_t}$ Ornstein-Uhlenbeck equation
- Electrostatic energy: $\displaystyle \langle E \rangle = \frac{\langle Q^2 \rangle}{2C} = \frac{1}{2} k_B T$
- Fluctuating current: $\delta I_t = \sigma \xi_t$

$$\Rightarrow \quad \sigma^2 = \frac{2k_BT}{R}, \qquad \sigma = \sqrt{\frac{2k_BT}{R}}$$

$$\Rightarrow \quad \sigma \propto \sqrt{T}$$

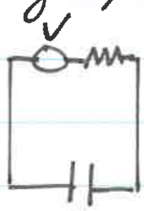
- Thermal electric noise
- Nyquist noise
- $\sim$ nV at room temperature

- Current source



δI_t

- Voltage source



$\delta V_t = R\, \delta I_t$

5.10 Geometric BM and Finance

Jacobs Sec. 5.2

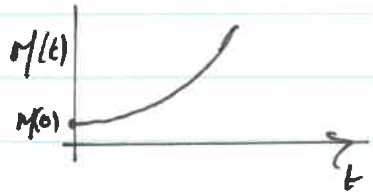
- Interest rate: M = amount of money

$$\Delta M = k \, M \, \Delta t$$

k: interest rate return; Δt: Period

$$\Rightarrow M(N\Delta t) = (1+k)^N M(0)$$ compounded interest

- Continuous interest: $dM = r\, M\, dt$



$$\Rightarrow M(t) = M(0)\, e^{rt}$$

$M(0)$: initial sum

- Risk free: Money goes up deterministically

- Stock market:
  - S_t = price of some asset at time t
  - "Mean" return: $dS_{\text{risk free}} = \mu\, S_t\, dt$
  - Random variations: $dS_{\text{random}} = \sigma\, S_t\, dW_t$

σ: volatility

Gaussian price movement proportional to asset

$$\Rightarrow dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$$
$$= (\mu\, dt + \sigma\, dW_t)\, S_t$$

drifted BM

- Solution:

$$S_t = S_0\, e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$