Chapter 4: Brownian motion

GS Sec. 13.2, Jacobs Chap. 3

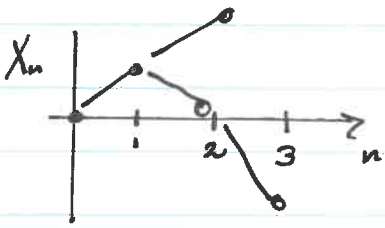
4.1 Simple random walk (Bernoulli RW)

- State: $S_i \in \mathbb{Z}$ displacement
- Evolution: $S_0 = 0$

$$S_n = S_{n-1} + X_n \qquad X_i \sim \text{Bern}(\pm 1, p)$$

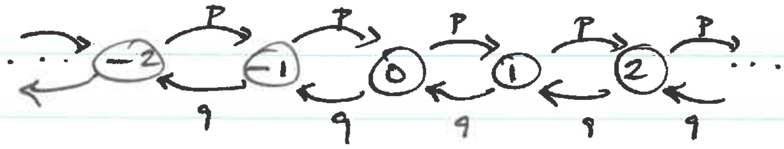
$$P(X_i = +1) = p$$
$$P(X_i = -1) = 1-p = q$$

$$S_0 = 0$$
$$S_1 = S_0 + X_1 = X_1$$
$$S_2 = S_1 + X_2 = X_1 + X_2$$
$$\vdots$$
$$S_n = \sum_{i=1}^{n} X_i \qquad X_i \sim \text{Bern} \ \underset{=}{\text{iid}}$$



- Mean displacement: $\overline{S_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$

- Markov representation:



$q = 1-p$

no self loops

$$S_n = S_{n-1} + X_n$$

displacement ± 1
jump

$$(\Pi_{ij}) = \begin{pmatrix} 0 & p & & 0 \\ q & \ddots & \ddots & \\ & \ddots & \ddots & \\ 0 & & & \end{pmatrix}$$

$\infty \times \infty$ matrix

- Properties:
  - $E[S_n] = n\,E[X] = n(p-(1-p)) = n(2p-1)$ $= 0$ if $p = \frac{1}{2}$
  - $E[\overline{S_n}] = 2p-1$
  - $\text{var}(S_n) = n\,\text{var}(X) = n\,4p(1-p)$
  - $\text{var}(\overline{S_n}) = 4p(1-p)/n$

· Simulation

```
n = 100
p = 0.7
s = zeros(n+1)
s[1] = 0
for i = 1:n
    s[i+1] = s[i] + berndisp(p)
end
plot([0:n], s)

function x = berndisp(p)
    if rand() < p
        x = +1
    else
        x = -1
    end
    x
end
```

· Note : Filling vector vs appending

```
s = [0]
for i = 1:n
    s = [s berndisp(p)]
end
```

or

```
for i in range(n):
    S.append(berndisp(p))
```

· Note : cumsum

```
xvec = [berndisp(p) for i in range(n)]
S = np.cumsum(xvec)
```

4.2 Gaussian random walk

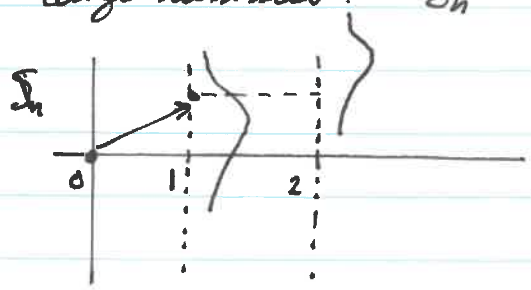
- Displacement:

$$S_n = \sum_{i=1}^{n} X_i \qquad X_i \sim N(\mu, \sigma^2)$$

$$= S_{n-1} + X_n \qquad S_0 = 0$$

- Properties:
  - $E[S_n] = n\mu$
  - $E[\bar{S}_n] = \mu$
  - $Var(S_n) = n\, Var(X) = n\sigma^2$
  - $Var(\bar{S}_n) = \frac{1}{n} Var(X) = \sigma^2/n$ $\searrow 0$ as $n \to \infty$

- Law of large numbers: $\bar{S}_n \xrightarrow{n\to\infty} \mu$ in probability as $n \to \infty$.



- Distribution:

$$p(S_n = s) = \frac{e^{-(s-n\mu)^2/2n\sigma^2}}{\sqrt{2\pi n\sigma^2}}$$

Sum of Gaussians is Gaussian

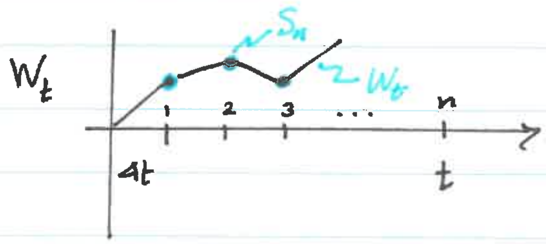
$$S_n \sim N(n\mu, n\sigma^2)$$

$$p(\bar{S}_n = \bar{s}) = \sqrt{\frac{n}{2\pi\sigma^2}}\; e^{-n\frac{(\bar{s}-\mu)^2}{2\sigma^2}}$$

- Simulation:

```
...
for i = 1:n
    s[i+1] = s[i] + randn()*sigma + mu
end
```

4.3 Wiener process (Brownian motion)



$n = \frac{t}{\Delta t}$, $\underline{W_0 = 0}$

· Discrete-time RW: $S_n = \sum_{i=1}^{n} X_i$ $\quad X_i \sim \mathcal{N}(0, \Delta t) = \sqrt{\Delta t}\, \mathcal{N}(0,1)$

· Continuous-time interpolation: $W_t = \lim_{\substack{\Delta t \to 0 \\ n \to \infty}} S_{t/\Delta t}$ $\quad$ piecewise constant or linear interpolation

· Properties:

· $E[W_t] = \lim_{\substack{\Delta t \to 0 \\ n \to \infty}} E[S_{\frac{t}{\Delta t}}] = 0$

· $\mathrm{var}(W_t) = \lim_{\substack{\Delta t \to 0 \\ n \to \infty}} n\, \mathrm{var}(X_i) = \lim_{\substack{\Delta t \to 0 \\ n \to \infty}} n \Delta t = t$

· Increments: $\Delta W_t = W_t - W_{t-\Delta t} = X_n \sim \mathcal{N}(0, \Delta t)$ independent

· W_t continuous but not differentiable (with probability 1)

· Distribution: $W_t \sim \mathcal{N}(0, t)$

$$p(W_t = w) = p(w,t) = \frac{e^{-w^2/2t}}{\sqrt{2\pi t}}$$

$$p(w,0) = \delta(w)$$

· Diffusion equation: $\frac{\partial p(w,t)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial w^2} p(w,t)$ $\quad$ ~ Laplacian

By direct substitution

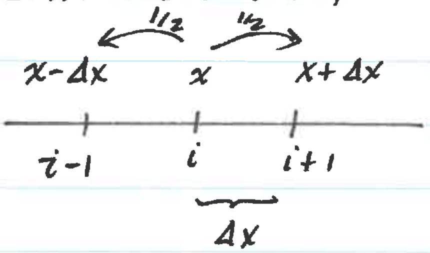
$\partial_t p = \frac{1}{2} \Delta p$ $\quad$ Heat equation

· Simulation: cf CW4

...

```
for i=1:n
    x[i+1] = x[i] + √dt * randn()
end
```

4.4 Continuum limit (Kramers-Moyal expansion)



Simple RW

- jump in time Δt
- Steps: $\pm \Delta x$
- $p = q = \frac{1}{2}$

· Distribution: $P(x,t) = P(X_t = i\,\Delta x)$ $x = i\,\Delta t$, $t = n\,\Delta t$

· Master equation:

$$\underbrace{P(x,t+\Delta t) - P(x,t)}_{\frac{d}{dt}p(x,t)} = \underbrace{\tfrac{1}{2}\,p(x-\Delta x,t)}_{\text{in }-} + \underbrace{\tfrac{1}{2}\,p(x+\Delta x,t)}_{\text{in }+} \underbrace{- p(x,t)}_{\text{out}}$$

$$\Rightarrow \quad p(x,t+\Delta t) = \tfrac{1}{2}\,p(x-\Delta x,t) + \tfrac{1}{2}\,p(x+\Delta x,t)$$ Balance equation

· Continuum limit: $\Delta t \to 0$, $\Delta x \to 0$

$$P(x,t+\Delta t) = p(x,t) + \frac{\partial p(x,t)}{\partial t}\,\Delta t + O(\Delta t^2)$$

$$p(x\pm\Delta x,t) = p(x,t) \pm \frac{\partial p(x,t)}{\partial x}\,\cancel{\Delta t} + \frac{\partial^2 p(x,t)}{\partial x^2}\,\frac{\Delta x^2}{2} + O(\Delta x^3)$$

Put in balance equation to get

$$\Delta t\,\frac{\partial p}{\partial t} = \frac{1}{2}\,\Delta x^2\,\frac{\partial^2 p}{\partial x^2}$$

or

$$\frac{\partial p}{\partial t} = \frac{D}{2}\,\frac{\partial^2 p}{\partial x^2} \qquad D = \lim_{\substack{\Delta x \to 0\\ \Delta t \to 0}} \frac{\Delta x^2}{\Delta t} = \frac{d}{dt}E[X_t^2] = 1$$ for BM

· Note: In real, physical Brownian motion,

$$D = \frac{RT}{N_a\,6\pi\,\eta\,a}$$

T: Temperature
N_a: Avogadro's number
η: Stokes friction
a: face area
R: Gas/liquid constant

4.5 Applications

4.5.1 Scaled BM

GS p519

$X_t = \sigma W_t \rightarrow X_t \sim \mathcal{N}(0, \sigma^2 t)$

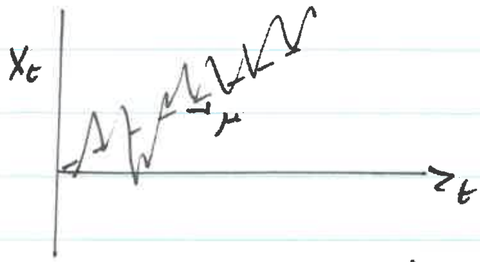
noise amplitude

4.5.2 Drifted BM

GS p520

$X_t = \mu t + \sigma W_t$

drifted noise

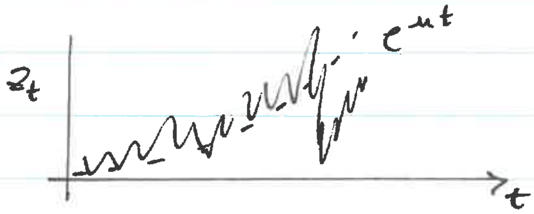


↳ Biased random walk, cf CW4. $X_t \sim \mathcal{N}(\mu t, \sigma^2 t)$

4.5.3 Geometric BM

GS 13.10 and Chap. 5

$Z_t = Z_0 e^{X_t}$



↳ Chap. 5.

4.5.4 2D BM

$\vec{R}_t = (X_t, Y_t)$

$X_t \sim \mathcal{N}(0, t)$
$Y_t \sim \mathcal{N}(0, t)$ independent