Universiteit Stellenbosch / *Stellenbosch University*
Toegepaste Wiskunde / *Applied Mathematics* **B252**
Assessering 1 / *Assessment 1*: 2017-09-01

STUDENTENOMMER / *STUDENT NUMBER*

INGENIEURSWESE
ENGINEERING

NEEM KENNIS VAN DIE INLIGTING HIERONDER /
TAKE NOTE OF THE INFORMATION BELOW

1. **Tydsduur: 2 Ure (120 Minute).** — **Duration: 2 Hours (120 Minutes).**
2. **Volpunte: 50.** — **Full Marks: 50.**
3. **Eksaminator: Dr Hanno Coetzer** — **Examiner: Dr Hanno Coetzer**
4. **Interne moderator: Dr Nick Hale** — **Internal moderator: Dr Nick Hale**

5. Beantwoord al die vrae. — Answer all the questions.
6. Vul die voorblad hierbo, ten volle in. — Complete in full the front cover above.
7. Sakrekenaars soos voorgeskryf vir die eerste twee jaar van die BIng mag gebruik word. — Pocket calculators prescribed for the first two years of the BEng may be used.
8. Hulpmiddels soos blanko papier, boeke, geskrifte en elektroniese apparaat (behalwe vir jou voorgeskrewe sakrekenaar en tekenapparaat) word nie in die toetslokaal toegelaat nie. — Supporting material like blank paper, books, notes and electronic equipment (except for your prescribed pocket calculator and drawing equipment) is not allowed in the test room.
9. Laatkommers kry nie addisionele skryftyd nie. — Latecomers are not allowed extra time.
10. Studente word nie toegelaat om die toetslokaal binne die eerste 45 minute van die toetssessie te verlaat nie. — Students are not allowed to leave the test room within the first 45 minutes of the test.
11. Kommunikasie tussen kandidate word nie in die toetslokaal toegelaat nie. — Communication between candidates is not allowed in the test room.
12. Hierdie vraag / antwoordboek, moet aan ’n opsiener oorhandig word voordat u die toetslokaal verlaat. U mag onder geen omstandighede hierdie boek met u saamneem uit die toetslokaal nie: — Before leaving the test room you must hand to an invigilator this question / answer book. You may under no circumstances take this book with you from the test room:

Kantoorgebruik *Office Use*	
1	/6
2	/8
3	/10
4	/10
5	/6
6	/10
Tot:	**/50**

LET OP: Die formules hieronder kan enige plek in die vraestel sonder bewys gebruik word – meer formules word in die probleemstellings verskaf.

PLEASE NOTE: The formulas below may be used anywhere in the paper without proof – more formulas are supplied in the problem statements.

$$\frac{d^2X}{dx^2} = \lambda^2 X \qquad \Longrightarrow \qquad X(x) = c_1 \cosh \lambda x + c_2 \sinh \lambda x$$

$$\frac{d^2X}{dx^2} = -\lambda^2 X \qquad \Longrightarrow \qquad X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$$

VRAAG 1 (Punte: 6)

'n Voorwerp met 'n massa van 5 kg word 'n aanvanklike afwaartse snelheid van 50 m/s gegee en dan toegelaat om onder die inwerking van swaartekrag te val.

Neem $g = 9.81\,\mathrm{m/s^2}$ en **aanvaar dat die lugweerstandskrag direk eweredig aan die spoed is**, met eweredigheidskonstante 10 kg/s.

Stel 'n differensiaalvergelyking (DV) op wat hierdie situasie modelleer. Los dan die DV op en verkry die snelheid van die voorwerp as 'n funksie van tyd. Bereik die voorwerp 'n limietsnelheid? Indien wel, bereken dit.

QUESTION 1 (Punte: 6)

An object with a mass of 5 kg is given an initial downward velocity of 50 m/s and is then allowed to fall under the influence of gravity.

Take $g = 9.81\,\mathrm{m/s^2}$ and **assume that the force due to air resistance is proportional to the speed**, with constant of proportionality 10 kg/s.

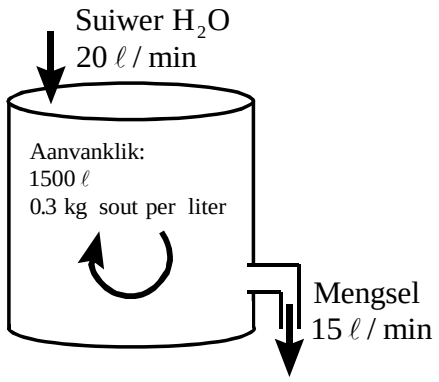
Set up a differential equation (DE) that models this situation. Then solve the DE and obtain the velocity of the object as a function of time. Does the object reach a limiting velocity? If so, calculate it.

Berekeninge vir Vraag 1 / Calculations for Question 1

Berekeninge vir Vraag 1 (vervolg) / Calculations for Question 1 (continued)

VRAAG 2 (Punte: 8)

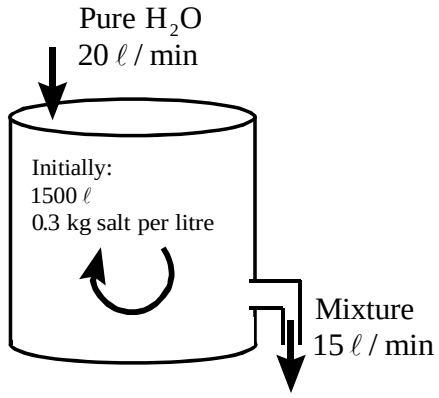
'n Tenk bevat aanvanklik 1500 liter soutwater met 'n konsentrasie van 0.3 kg sout per liter (sien figuur). Suiwer water vloei nou die tenk binne teen 'n konstante tempo van 20 liter per minuut. Die mengsel in die tenk word goed geroer, en vloei na buite teen 'n konstante tempo van 15 liter per minuut.



(a) Laat $x = x(t)$ die **massa** sout in die tenk op tyd t voorstel, en toon aan dat

QUESTION 2 (Marks: 8)

A tank initially contains 1500 litres of brine with a concentration of 0.3 kg salt per litre (see figure). Pure water now enters the tank at a constant rate of 20 litres per minute. The mixture is stirred well, and exits the tank at a constant rate of 15 litres per minute.



(a) Let $x = x(t)$ be the **mass** of the salt in the tank at time t, and show that

$$\frac{dx}{dt} + \frac{3x}{t+300} = 0 \quad \text{waar/}\textit{where} \quad x(0) = 450.$$

Berekeninge vir Vraag 2 (a) [Punte: 2] / Calculations for Question 2 (a) [Marks: 2]

(b) Los die aanvangswaardeprobleem in **(a)** op. Na hoeveel minute sal die **konsentrasie** sout in die tenk gelyk wees aan 0.15 kg/liter? Aanvaar dat die tenk nie oorloop nie.

(b) Solve the initial-value problem in **(a)**. After how many minutes will the **concentration** of salt in the tank be equal to 0.15 kg/litre? Assume that the tank does not overflow.

Berekeninge vir Vraag 2 (b) [Punte: 6] / Calculations for Question 2 (b) [Marks: 6]

Tydsduur/Duration =

VRAAG 3 (Punte: 10)

Beskou die volgende aanvangswaardeprobleem binne die konteks van bevolkingsdinamika:

$$\frac{dP}{dt} = P\,(a - bP) - h, \qquad P(0) = P_0$$

Bespreek bogenoemde probleem kortliks. Wat is die betekenis van a, b en h?

Neem nou $a = 5$, $b = 1$ en $h = 25/4$. Bespreek die langtermyn gedrag van die oplossing (wanneer $t \to \infty$) vir verskillende waardes van P_0. Maak ruim van sketse gebruik. Is daar waardes van P_0 waarvoor die bevolking sal uitsterf en indien wel, hoe lank sal dit neem?

QUESTION 3 (Marks: 10)

Consider the following initial-value problem within the context of population dynamics:

Briefly discuss the above-mentioned problem. What is the meaning of a, b and h?

Now take $a = 5$, $b = 1$, and $h = 25/4$. Discuss the long-term behaviour of the solution (when $t \to \infty$) for different values of P_0. Make ample use of sketches. Are there values of P_0 for which the population will become extinct and if so, how long will it take?

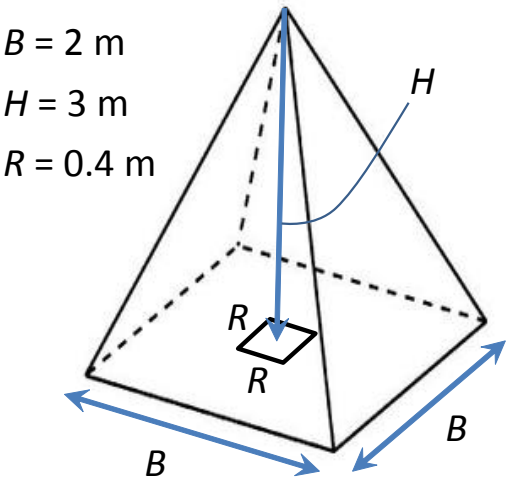
Berekeninge vir Vraag 3 / Calculations for Question 3

Berekeninge vir Vraag 3 (vervolg) / Calculations for Question 3 (continued)

VRAAG 4 (Punte: 10)

(a) Beskou 'n tenk in die vorm van 'n piramide wat 'n vertikale hoogte van 3 m het en deur 'n gat in sy vierkantige bodem (met 'n sylengte van 2 m) dreineer (soos in die figuur). Aanvaar dat die gat ook vierkantig is (met 'n sylengte van 0.4 m). Die tenk is aanvanklik tot 'n diepte van y_0 meter met water gevul.

Aanvaar verder dat, wanneer die diepte van die water y is, 'n waterdruppel by die gat uitspuit teen 'n snelheid van $v = c\sqrt{2gy}$ (Torricelli se wet met 'n verliesfaktor). Kies $g = 9.81\,\mathrm{m/s^2}$ en $c = 0.9$.



Dit is nie nodig om die aanvangswaardeprobleem op te los nie.

QUESTION 4 (Marks: 10)

(a) Consider a pyramid-shaped tank (as shown in the figure) with a vertical height of 3 m that drains through a hole in its square-shaped bottom (with a side length of 2 m). Assume that the hole is also square-shaped (with a side-length of 0.4 m). The tank is initially filled with water to a depth of y_0 meters.

Further assume that, when the depth of the water is y, a water droplet exits the tank with a velocity of $v = c\sqrt{2gy}$ (Torricelli's law with a loss factor). Choose $g = 9.81\,\mathrm{m/s^2}$ and $c = 0.9$.

Toon vanuit eerste beginsels aan dat hierdie situasie deur die volgende aanvangswaardeprobleem gemodelleer word, waar jy self $f(y)$ moet vind:

$$\frac{dy}{dt} = f(y), \qquad y(0) = y_0.$$

Show from first principles that this situation is modelled by the following initial-value problem, where you have to find $f(y)$ yourself:

$$\frac{dy}{dt} = f(y), \qquad y(0) = y_0.$$

It is not necessary to solve the initial-value problem.

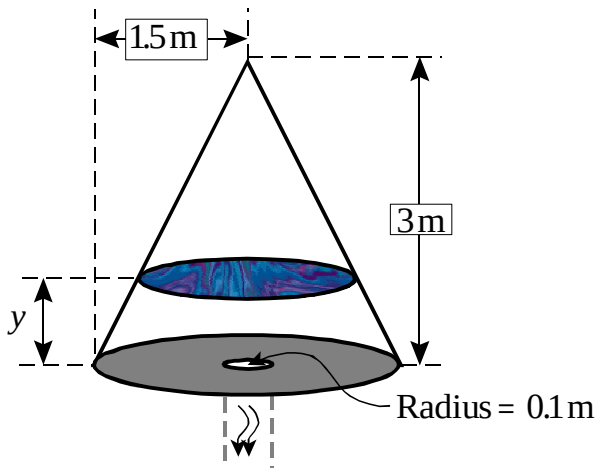
Berekeninge vir Vraag 4 (a) [Punte: 5] / Calculations for Question 4 (a) [Marks: 5]

Berekeninge vir Vraag 4 (a) (vervolg) / Calculations for Question 4 (a) (continued)

$f(y) =$

(b) Beskou 'n keëelvormige tenk met 'n vertikale hoogte van 3 m wat deur 'n gat in sy sirkelvormige bodem (met 'n radius van 1.5 m) dreineer (soos in die figuur). Aanvaar dat die gat ook sirkelvormig is (met 'n radius van 0.1 m). Die tenk is aanvanklik tot 'n diepte van 1 m met water gevul.

Aanvaar verder dat, wanneer die diepte van die water y is, 'n waterdruppel by die gaatjie uitspuit met 'n snelheid van $v = \sqrt{2gy}$ (Toricelli se wet, sonder 'n verliesfaktor) en dat $g = 9.81\,\mathrm{m/s^2}$.



Los die aanvangswaardeprobleem op en bepaal in hoeveel sekondes die tenk sal dreineer.

(b) Consider a cone-shaped tank (as shown in the figure) with a vertical height of 3 m that drains through a hole in its circular bottom (with a radius of 1.5 m). Assume that the hole is also circular (with a radius of 0.1 m). The tank is initially filled with water to a depth of 1 m.

Further assume that, when the depth of the water is y, a water droplet exits the tank with a velocity of $v = \sqrt{2gy}$ (Torricelli's law, without a loss factor) and that $g = 9.81\,\mathrm{m/s^2}$.

Aanvaar dat hierdie situasie deur die volgende aanvangswaardeprobleem gemodelleer word:

$$(3-y)^2\,\frac{dy}{dt} = -0.177\,y^{1/2}, \qquad y(0) = 1.$$

Assume that this situation is modelled by the following initial-value problem:

$$(3-y)^2\,\frac{dy}{dt} = -0.177\,y^{1/2}, \qquad y(0) = 1.$$

Solve the initial-value problem and determine in how many seconds the tank will drain.

Berekeninge vir Vraag 4 (b) [Punte: 5] / Calculations for Question 4 (b) [Marks: 5]

Berekeninge vir Vraag 4 (b) (vervolg) / Calculations for Question 4 (b) (continued)

Tydsduur / Duration =

VRAAG 5 (Punte: 6)

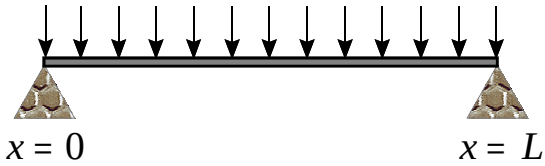
Beskou die balkvergelyking (links onder):

$$\boxed{EI\,\frac{d^4y}{dx^4} = w(x),\ x \in [0, L]}$$

Aanvaar sonder bewys dat:

QUESTION 5 (Marks: 6)

Consider the beam equation (below left):



Assume without proof that:

Endpunt van Balk	Randvoorwaarde/Boundary Condition	End of the Beam
Ingeklem	$\boxed{y = 0, \quad y' = 0}$	Embedded
Vry	$\boxed{y'' = 0, \quad y''' = 0}$	Free
Eenvoudig Ondersteun	$\boxed{y = 0, \quad y'' = 0}$	Simply Supported

Aanvaar verder dat $w(x) = w_0$, waar $x \in [0, L]$ en w_0 'n konstante is. Die balk is aan die linkerkant en regterkant eenvoudig ondersteun (sien die figuur regs bo).

Vind nou 'n uitdrukking vir die maksimum uitwyking van die balk in terme van w_0, L, E en I.

Further assume that $w(x) = w_0$, where $x \in [0, L]$ and w_0 is a constant. The beam is simply supported on the left and right (see the figure above right).

Now obtain an expression for the maximum deflection of the beam in terms of w_0, L, E and I.

Berekeninge vir Vraag 5 / Calculations for Question 5

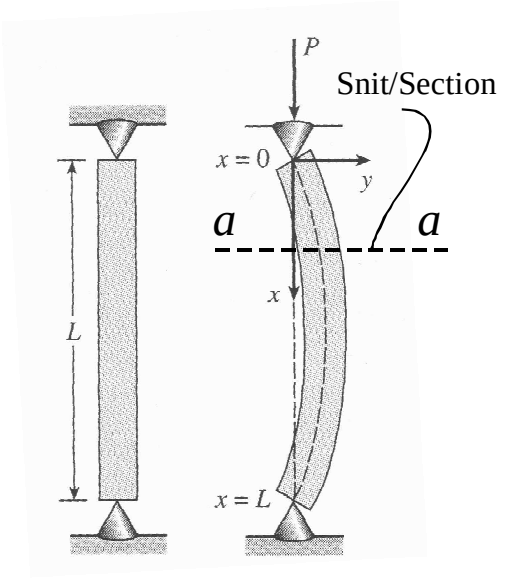
Berekeninge vir Vraag 5 (vervolg) / Calculations for Question 5 (continued)

$y_{\max} =$

VRAAG 6 (10 punte)

Beskou die probleem van knikking in 'n vertikale kolom (sien die figuur links onder) en aanvaar dat die Euler-Bernoulli wet geld,

$$\boxed{M = EIK}, \quad \text{waar/}\textit{where} \quad K = \frac{d^2y/dx^2}{\left(1 + (\frac{dy}{dx})^2\right)^{3/2}}.$$



QUESTION 6 (10 marks)

Consider the problem of buckling in a vertical column (see the figure below left) and assume that the Euler-Bernoulli law applies,

(a) Stel duidelik watter aannames en benaderings gemaak word, en toon vanuit eerste beginsels aan dat hierdie situasie deur die volgende randwaardeprobleem gemodelleer word,

$$EI\,\frac{d^2y}{dx^2} + Py = 0, \qquad y(0) = 0, \quad y(L) = 0.$$

(a) *State clearly what assumptions and approximations are made, and show from first principles that this situation is modelled by the following boundary-value problem,*

$$EI\,\frac{d^2y}{dx^2} + Py = 0, \qquad y(0) = 0, \quad y(L) = 0.$$

Berekeninge vir Vraag 6 (a) [3 punte] / Calculations for Question 6 (a) [3 marks]

(b) Herlei nou formules vir die kritieke belastings. Toon alle berekeninge.

(b) Now derive formulas for the critical loads. Show all your calculations.

Berekeninge vir Vraag 6 (b) [5 punte] / Calculations for Question 6 (b) [5 marks]

(c) Gee die formule vir die Euler belasting.

(c) Give the formula for the Euler load.

Berekeninge vir Vraag 6 (c) [1 punt] / Calculations for Question 6 (c) [1 mark]

(d) Gee die formule vir die tweede knikkingsmodus, en skets dit.

(d) Give the formula for the second buckling mode, and sketch it.

Berekeninge vir Vraag 6 (d) [1 punt] / Calculations for Question 6 (d) [1 mark]