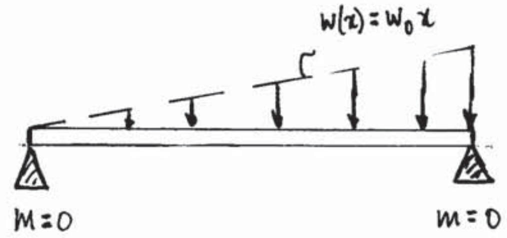
QUESTION 1

Ex 3.9; Nr 5(a)



$\Rightarrow y(0) = 0$ $\qquad$ $\Rightarrow y(L) = 0$

$y''(0) = 0$ $\qquad$ $y''(L) = 0$

$$\frac{d^4y}{dx^4} = \frac{w_0}{EI}x = Ax \quad ; \quad A = \frac{w_0}{EI}$$

$$y'''(x) = \frac{1}{2}Ax^2 + B$$

$$y''(x) = \frac{1}{6}Ax^3 + Bx + C$$

$$y''(0) = C = 0$$

$$y''(L) = \frac{1}{6}AL^3 + BL = 0 \quad \sim (*)$$

$$y'(x) = \frac{1}{24}Ax^4 + \frac{1}{2}Bx^2 + D$$

$$y(x) = \frac{1}{120}Ax^5 + \frac{1}{6}Bx^3 + Dx + E$$

$$y(0) = E = 0$$

$$y(L) = \frac{1}{120}AL^5 + \frac{1}{6}BL^3 + DL = 0 \quad \sim (**)$$

From $(*)$: $\quad B = -\frac{1}{6}AL^2$

Into $(**)$: $\quad D = -\frac{1}{120}AL^4 - \frac{1}{6}\left(-\frac{1}{6}AL^2\right)L^2$

$$= \frac{7}{360}AL^4$$

$$\Rightarrow y(x) = \frac{1}{120}Ax^5 + \frac{1}{6}\left(-\frac{1}{6}AL^2\right)x^3 + \frac{7}{360}AL^4x$$

$$= \frac{Ax}{360}\left(3x^4 - 10L^2x^2 + 7L^4\right)$$

$$A = \frac{w_0}{EI}$$

Ex 3.9; Nr 5(b) and (c)

$y = y_{max}$ when $y'(x) = 0$

$$\Rightarrow \frac{1}{24}Ax^4 - \frac{1}{12}AL^2x^2 + \frac{7}{360}AL^4 = 0$$

$$\Rightarrow \frac{A}{360}\left(15x^4 - 30L^2x^2 + 7L^4\right) = 0$$

Let $Z = x^2 \Rightarrow y'(x) = 0$ if:

$$15Z^2 - 30L^2Z + 7L^4 = 0$$

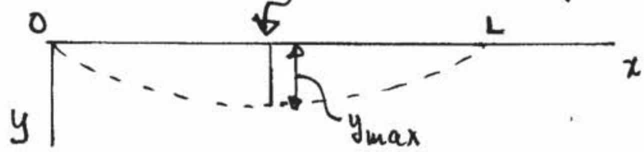
$$\Rightarrow Z = \frac{30L^2 \pm \sqrt{(30L^2)^2 - 4(15)(7L^4)}}{2(15)}$$

$$= \frac{30L^2 \pm 30L^2\sqrt{1 - 7/15}}{30}$$

$$= L^2\left(1 \pm \sqrt{8/15}\right)$$

$$= 0{,}2697L^2 \; ; \; 1{,}7303L^2$$

$$\Rightarrow x = \sqrt{Z} = 0{,}5193L \; ; \; \underbrace{1{,}3154L}_{\text{Not physical}}$$



$$y_{max} = y(0{,}5193L) = 0{,}006522\frac{w_0}{EI}L^5$$

When $w_0 = 36EI$ and $L = 1$, then

$$y_{max} = 0{,}2348$$

QUESTION 2

(a)

$$y'' + 4y = 0$$

General solution: $y(t) = c_1 \cos 2t + c_2 \sin 2t$

$$y' = -2c_1 \sin 2t + 2c_2 \cos 2t$$

$y(0) = 0 \Rightarrow c_1 = 0$

$y'(0) = -10 \Rightarrow -10 = 2c_2 \Rightarrow c_2 = -5$

$\Rightarrow$ Specific solution: $y(t) = -5 \sin 2t$

(b) $y'' + 10y' + 16y = 0$

Test solution: $y = e^{mt}$

$\Rightarrow m^2 + 10m + 16 = 0$

$\Rightarrow (m+8)(m+2) = 0 \Rightarrow m_1 = -8;\ m_2 = -2$

$\Rightarrow$ General solution: $y(t) = c_1 e^{-8t} + c_2 e^{-2t}$

$y(0) = 1 \Rightarrow c_1 + c_2 = 1 \sim ①$

$y'(0) = -12 \Rightarrow -8c_1 - 2c_2 = -12 \sim ②$

From ① and ② $\rightarrow c_1 = \frac{5}{3};\ c_2 = -\frac{2}{3}$

Specific solution

$\Rightarrow y(t) = \frac{5}{3} e^{-8t} - \frac{2}{3} e^{-2t}$

(c) $y'' + 2y' + 2y = 2\sqrt{2} \cos(\sqrt{2}t)$

$y(t) = y_{\text{homogeneous}} + y_{\text{particular}}$

$\quad = y_h + y_p$

Homogeneous problem: $y'' + 2y' + 2y = 0$

Test solution: $y = e^{mt}$

$\Rightarrow m^2 + 2m + 2 = 0$

$\Rightarrow m_1 = -1 + i;\ m_2 = -1 - i$

$\Rightarrow y_h = c_1 e^{(-1+i)t} + c_2 e^{(-1-i)t}$

$\quad = e^{-t}(c_1 e^{it} + c_2 e^{-it})$

$\quad = e^{t}(d_1 \cos t + d_2 \sin t)$

Particular problem: $y'' + 2y' + 2y = 2\sqrt{2} \cos(\sqrt{2}t) \sim (*)$

Let $y_p = A \cos(\sqrt{2}t) + B \sin(\sqrt{2}t)$

$y_p' = -\sqrt{2} A \sin(\sqrt{2}t) + \sqrt{2} B \cos(\sqrt{2}t)$

$y_p'' = -2A \cos(\sqrt{2}t) - 2B \sin(\sqrt{2}t)$

Substitute into $(*)$:

$-2A \cos(\sqrt{2}t) - 2B \sin(\sqrt{2}t) - 2\sqrt{2} A \sin(\sqrt{2}t)$
$+ 2\sqrt{2} B \cos(\sqrt{2}t) + 2A \cos(\sqrt{2}t) + 2B \sin(\sqrt{2}t) = 2\sqrt{2} \cos(\sqrt{2}t)$

Compare sin-terms: $-2B - 2\sqrt{2}A + 2B = 0 \Rightarrow A = 0$

Compare cos-terms: $-2A + 2\sqrt{2}B + 2A = 2\sqrt{2} \Rightarrow B = 1$

$\Rightarrow y_p = \sin(\sqrt{2}t)$ General solution

$\Rightarrow y(t) = y_h + y_p = e^{-t}(d_1 \cos t + d_2 \sin t) + \sin(\sqrt{2}t)$

$y(0) = d_1 = 0 \Rightarrow y(t) = d_2 e^{-t} \sin t + \sin(\sqrt{2}t)$

$y'(t) = d_2[e^{-t} \cos t - e^{-t} \sin t] + \sqrt{2} \cos(\sqrt{2}t)$

$y'(0) = d_2 + \sqrt{2} = 0 \Rightarrow d_2 = -\sqrt{2}$

$\Rightarrow y(t) = -2e^{-t} \sin t + \sin(\sqrt{2}t)$

Specific solution

QUESTION 3

Ex 3.9; Nr 13

$$\frac{d^2y}{dx^2} + \lambda y = 0$$ ← Changed from textbook

$y'(0) = 0$; $y'(\pi) = 0$

For $\lambda = 0$: $\frac{d^2y}{dx^2} = 0$

$\Rightarrow \frac{dy}{dx} = c_1$

$y'(0) = 0 \Rightarrow c_1 = 0$

$y'(\pi) = 0 \Rightarrow c_1 = 0$

$\Rightarrow y(x) = c_2$ ← arbitrary constant

$\Rightarrow \lambda = 0$ is an eigenvalue

with $y = c$ the corresponding eigenfunction (constant $\neq 0$)

Note: $y = 0$ is trivial and can not be an eigenfunction, but $y = c$ is not trivial and can be an eigenfunction as long as $c \neq 0$

For $\lambda < 0$: Let $\lambda = -\alpha^2$ $\quad (\alpha \neq 0)$

$\Rightarrow \frac{d^2y}{dx^2} = \alpha^2 y$

$\Rightarrow y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$

$y' = c_1 \alpha \sinh \alpha x + c_2 \alpha \cosh \alpha x$

$y'(0) = c_2 \alpha = 0 \quad \Rightarrow c_2 = 0$

$y'(\pi) = c_1 \underbrace{\alpha \sinh \alpha\pi}_{\neq 0} = 0 \Rightarrow c_1 = 0$

$\Rightarrow y = 0$ (trivial solution)

For $\lambda > 0$: Let $\lambda = +\alpha^2$ $\quad (\alpha \neq 0)$

$\Rightarrow \frac{d^2y}{dx^2} = -\alpha^2 y$

$\Rightarrow y = c_1 \cos \alpha x + c_2 \sin \alpha x$

$y' = -c_1 \alpha \sin \alpha x + c_2 \alpha \cos \alpha x$

$y'(0) = c_2 \alpha = 0 \Rightarrow c_2 = 0$

$y'(\pi) = -c_1 \alpha \sin \alpha\pi = 0$

For non-trivial solutions: $\sin \alpha\pi = 0$

$\Rightarrow \alpha\pi = \pm\pi; \pm 2\pi; \pm 3\pi; \ldots$

$\Rightarrow \alpha = \pm 1; \pm 2; \pm 3; \ldots$

$\Rightarrow \lambda = \alpha^2 = 1, 4, 9, \ldots$

Eigenvalues, with eigenfunctions:

$y = a_1 \cos x; a_2 \cos 2x; a_3 \cos 3x; \ldots$

In summary (including $\lambda = 0$; $y = c$)

Eigenvalues

$$\left.\begin{aligned} \lambda_k &= k^2; \\ y_k &= a_k \cos kx \end{aligned}\right\} k = 0, 1, 2, 3, \ldots$$

included

Eigenfunctions

QUESTION 4

Ex 3.9; Nr 22

$$EI\frac{d^2y}{dx^2} + Py = P\delta$$

General solution:

$$y = c_1 \cos\sqrt{\frac{P}{EI}}\,x + c_2 \sin\sqrt{\frac{P}{EI}}\,x + \delta$$

Particular solution found by inspection

The column is imbedded at $x=0$, therefore the boundary conditions are:

$$y(0)=0 \quad \text{and} \quad y'(0)=0.$$

(a) When $\delta = 0$, we obtain from the boundary conditions that $c_1 = c_2 = 0$. Therefore $y = 0$.

(b) When $\delta \neq 0$, we obtain from the boundary conditions that $c_1 = -\delta$ and $c_2 = 0$. Therefore:

$$y = \delta\left(1 - \cos\sqrt{\frac{P}{EI}}\,x\right)$$

A further condition is that $y(L) = \delta$, therefore:

$$\delta = \delta\left(1 - \cos\sqrt{\frac{P}{EI}}\,L\right)$$

and

$$\cos\sqrt{\frac{P}{EI}}\,L = 0.$$

From this it follows that

$$\sqrt{\frac{P}{EI}}\,L = \frac{n\pi}{2}, \quad n = 1, 3, 5, \ldots$$

The Euler load (at $n=1$) is therefore given by

$$\sqrt{\frac{P_1}{EI}}\,L = \frac{\pi}{2}$$

Therefore

$$P_1 = \frac{1}{4}\left(\frac{\pi^2 EI}{L^2}\right)$$