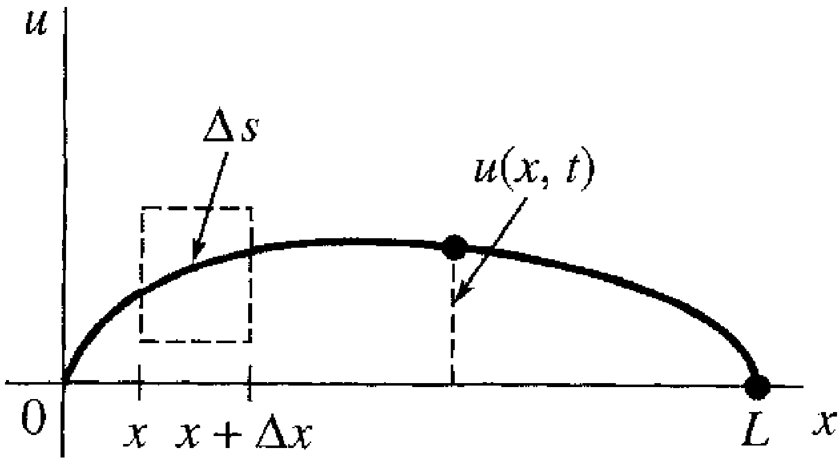
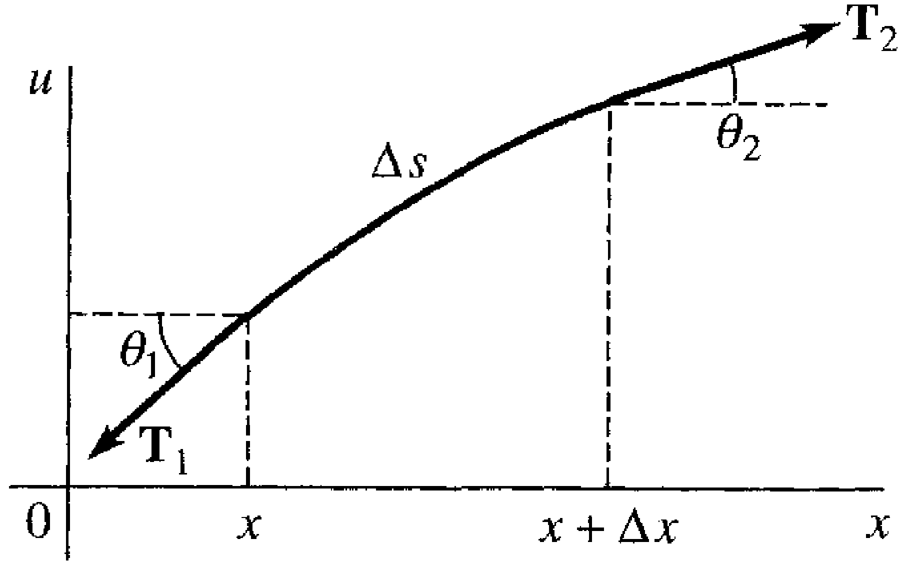
The wave equation $u_{tt} = a^2 u_{xx}$

<u>Derivation</u> *(page 694)*



(a) segment of string



(b) enlargement of segment

Assumptions

(1) The tension in the string is everywhere the same

$$T_1 = T_2$$

(2) The deflection is small
(3) String is uniform (mass/length (ρ) constant)
(4) The tension T is large $\Rightarrow$ gravity can be ignored

Newton's second law ($\Sigma F = ma$):

$$T_2 \sin\theta_2 - T_1 \sin\theta_1 = m\frac{\partial^2 u}{\partial t^2}$$

Small deflection $\Rightarrow$ ...

$$\sin\theta_1 \approx \tan\theta_1 \approx \frac{\partial u}{\partial x}(x,t)$$

$$\sin\theta_2 \approx \tan\theta_2 \approx \frac{\partial u}{\partial x}(x+\Delta x,t)$$

$$m = \rho\Delta x \quad \text{and} \quad T_1 = T_2 = T$$

$$\rho\Delta x\frac{\partial^2 u}{\partial t^2} = T\left(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right)$$

$$\frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} = \frac{\left(\frac{\partial u}{\partial x}(x+\Delta x, t) - \frac{\partial u}{\partial x}(x,t)\right)}{\Delta x}$$

If $\Delta x \to 0$ then

$$\frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2}$$

Let $\dfrac{T}{\rho} = a^2$ then $\boxed{u_{tt} = a^2 u_{xx}}$

Initial conditions:

$$\begin{aligned} u(x,0) &= f(x) \quad \text{(initial position)} \\ u_t(x,0) &= g(x) \quad \text{(initial velocity)} \end{aligned}$$

Boundary conditions:

$$\begin{aligned} u(0,t) &= 0 \quad \text{(left endpoint fixed)} \\ u(L,t) &= 0 \quad \text{(right endpoint fixed)} \end{aligned}$$

(Example) PDE: $u_{tt} = a^2 u_{xx}, \quad a^2 = \frac{T}{\rho}$

Initial conditions:

(1) $u(x, 0) = \sin(\pi x)$ **(2)** $u_t(x, 0) = 0, \; x \in [0, 1]$

Boundary conditions:

(1) $u(0, t) = 0$ **(2)** $u(1, t) = 0$

Solution: $u(x, t) = \cos(\pi a t) \sin(\pi x)$

It will be expected of you to obtain this solution in the next tutorial

Meanwhile, test the validity of the solution - does it satisfy the PDE, the boundary conditions, and the initial conditions!

Note that the period of the motion is given by

$$\frac{2\pi}{\pi a} = \frac{2}{a}$$

(interpret)

$$u(x,t) = \cos(\pi a t)\sin(\pi x)$$