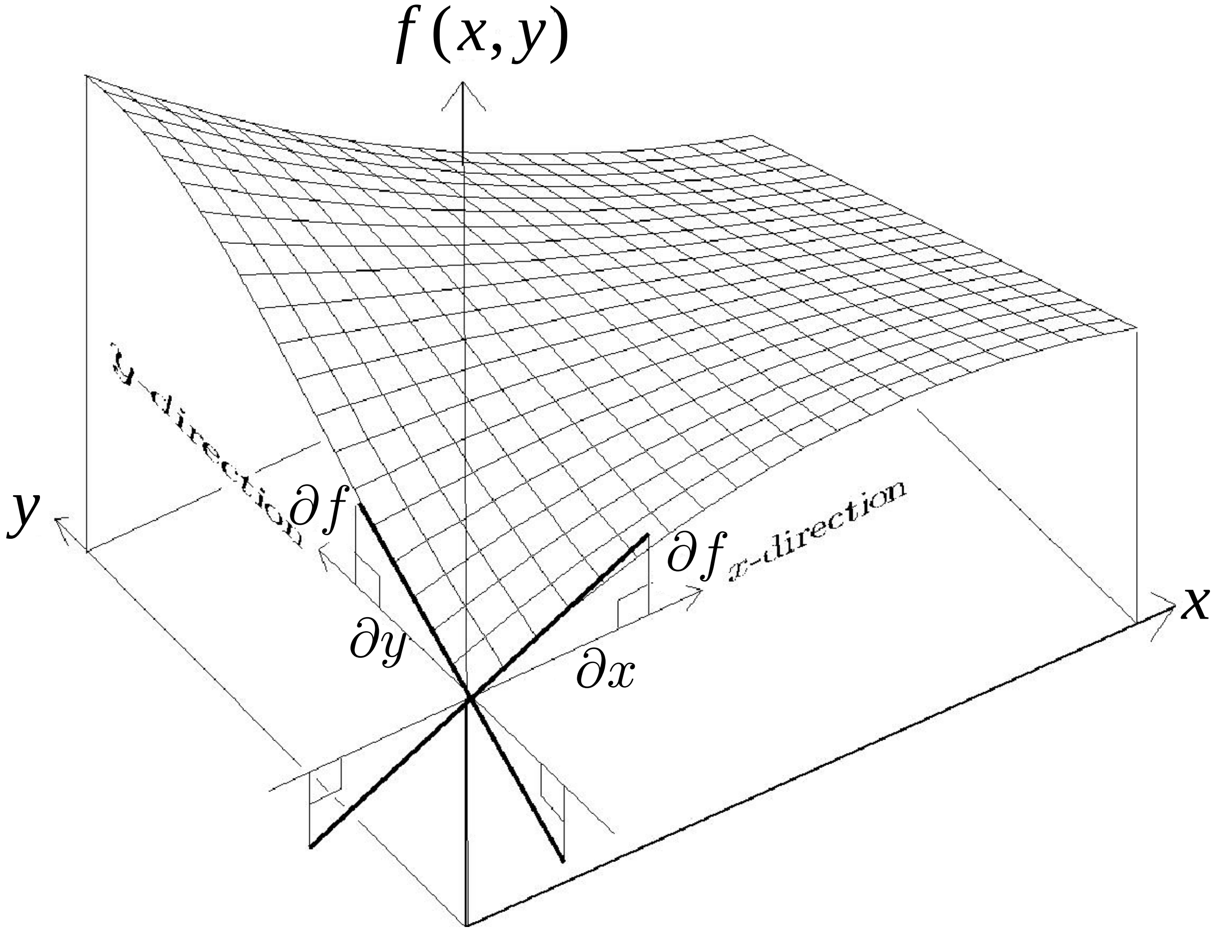
Parsiële afgeleides



Voorbeeld: $f(x,y) = x^2y^3 + x^5\cos y$

$$\frac{\partial f}{\partial x} = 2xy^3 + 5x^4\cos y;\ \frac{\partial f}{\partial y} = 3x^2y^2 - x^5\sin y$$

$$\frac{\partial^2 f}{\partial x^2} = 2y^3 + 20x^3\cos y;\ \frac{\partial^2 f}{\partial x\partial y} = 6xy^2 - 5x^4\sin y$$

Hfst 13: Parsiële DVs

Gewone DV: Net een onafhanklike veranderlike, bv.

$$y = y(x) \quad \text{of} \quad y = y(t)$$

Parsiële DV: Meer as een onafhanklike veranderlike, bv.

$$y = y(x, t) \quad \text{of} \quad u = u(x, y, z)$$

13.1: Skeibare PDVs *(bl 664)*

Mees algemene lineêre 2de orde PDV:

$$\boxed{A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G}$$

Die koëffisiënte,

$$A = A(x, y),\ B = B(x, y),\ \ldots,\ G = G(x, y),$$

is almal gegee (dikwels is A, B, ..., G konstant)

Homogeen as $G = 0$; Nie-homogeen as $G \neq 0$

Ons wil dus die oplossing $\boxed{u = u(x, y)}$ vind

Voorbeelde

Hitte- of diffusievergelyking (lineêr)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (c^2 \equiv \text{diffusie-konstante})$$

Golfvergelyking (lineêr) Golfvergelyking (lineêr)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (c \equiv \text{konstante (spoed)})$$

Laplace se vergelyking (lineêr)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Burger se vergelyking (nie-lineêr)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

Klassifikasie van PDVs

<u>Hiperbolies</u> as $B^2 - 4AC > 0$
<u>Parabolies</u> as $B^2 - 4AC = 0$
<u>Ellipties</u> as $B^2 - 4AC < 0$

Toon nou aan dat die hittevergelyking parabolies, die golfvergelyking hiperbolies en Laplace se vergelyking ellipties is

Produk-oplossings, skeiding van veranderlikes

Soek oplossings van die vorm: $\boxed{u(x,y) = X(x)Y(y)}$

$$\Rightarrow \frac{\partial u}{\partial x} = X'(x)Y(y) \qquad \frac{\partial u}{\partial y} = X(x)Y'(y)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)Y(y) \qquad \frac{\partial^2 u}{\partial y^2} = X(x)Y''(y)$$

$$\frac{\partial^2 u}{\partial x \partial y} = X'(x)Y'(y)$$

Stel in PDV $\Rightarrow$ skei sodoende PDV in 2 **gewone** DVs, een in X en een in Y!

Voorbeeld 1: *(Bl 692, Nr 4)* Gebruik skeiding van veranderlikes en vind (indien moontlik) produkoplossings vir die PDV:

$$u_x = u_y + u$$

Soek oplossings van die vorm: $\boxed{u(x,y) = X(x)Y(y)}$

$$\Rightarrow \quad u_x = X'Y \quad \text{en} \quad u_y = XY'$$

$$\Rightarrow \quad X'Y = XY' + XY$$

Deel nou regdeur met XY:

$$\Rightarrow \quad \frac{X'(x)}{X(x)} = \frac{Y'(y)}{Y(y)} + 1 \tag{1}$$

Let op dat die LK van (1) net afhanklik is van x, en dat die RK van (1) net afhanklik is van y. Vgl (1) moet egter geldig wees vir alle x **en** y. Dit is net moontlik as die LK **en** RK gelyk aan 'n konstante, sê λ, is! Ons noem λ die **skeidingskonstante**.

$$\Rightarrow \quad \frac{X'}{X} = \lambda \quad \text{en} \quad \frac{Y'}{Y} + 1 = \lambda$$

Ons het dus 1 PDV met 2 gewone DVs vervang!

Oplossing vir $\dfrac{X'}{X} = \lambda$: $\qquad X(x) = ce^{\lambda x}$

Oplossing vir $\dfrac{Y'}{Y} + 1 = \lambda$: $Y(y) = de^{(\lambda-1)y}$

Die produkoplossing word dus gegee deur,

$$u(x, y) = X(x)Y(y) = ae^{\lambda(x+y)-y},$$

vir alle konstantes a en λ.

Bevestig! $\quad u_x = \lambda\left(ae^{\lambda(x+y)-y}\right) = \lambda u$

$$u_y = (\lambda - 1)u = \lambda u - u$$

$$\text{RK} = u_y + u = (\lambda u - u) + u = \lambda u = u_x = \text{LK}$$

$\Rightarrow$ Produkoplossing bevredig die PDV!

Voorbeeld 2: Gebruik skeiding van veranderlikes en vind (indien moontlik) produkoplossings vir die PDV:

$$u_{xx} + u_{yy} = 0$$

Soek oplossings van die vorm: $\boxed{u(x,y) = X(x)Y(y)}$

$$\Rightarrow \quad u_{xx} = X''Y \quad \text{en} \quad u_{yy} = XY''$$

$$\Rightarrow \quad X''Y + XY'' = 0$$

Deel nou regdeur met XY:

$$\Rightarrow \ \frac{X''}{X} + \frac{Y''}{Y} = 0 \ \Rightarrow \ \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \qquad (2)$$

Let op dat die LK van (2) net afhanklik is van x, en dat die RK van (2) net afhanklik is van y. Vgl (2) moet egter geldig wees vir alle x **en** y. Dit is net moontlik as die LK **en** RK gelyk aan 'n konstante, sê $\pm\lambda^2$, is! (Die kwadraat is gerieflik.)

Vir skeidingskonstante $\boxed{\lambda = 0}$

$$\Rightarrow \quad X'' = 0 \quad \text{en} \quad Y'' = 0$$

$$\Rightarrow \quad X = ax + b \quad \text{en} \quad Y = cy + d$$

$$\Rightarrow \quad u(x, y) = (ax + b)(cy + d)$$

Bevestig nou dat dit die PDV bevredig

Vir skeidingskonstante $\boxed{+\lambda^2}$

$$\Rightarrow \quad X'' = +\lambda^2 X \quad \text{en} \quad Y'' = -\lambda^2 Y$$

$$\Rightarrow \quad X = a\cosh(\lambda x) + b\sinh(\lambda x) \quad \text{en}$$

$$Y = c\cos(\lambda y) + d\sin(\lambda y) \quad \text{dus...}$$

$$u(x, y) = [a\cosh(\lambda x) + b\sinh(\lambda x)]\,[c\cos(\lambda y) + d\sin(\lambda y)]$$

Bevestig nou dat dit die PDV bevredig

Vir skeidingskonstante $\boxed{-\lambda^2}$

$$\Rightarrow \quad X'' = -\lambda^2 X \quad \text{en} \quad Y'' = +\lambda^2 Y$$

$$\Rightarrow \quad X = a\cos(\lambda x) + b\sin(\lambda x) \quad \text{en}$$

$$Y = c\cosh(\lambda y) + d\sinh(\lambda y) \quad \text{dus...}$$

$$u(x,y) = [a\cos(\lambda x) + b\sin(\lambda x)]\,[c\cosh(\lambda y) + d\sinh(\lambda y)]$$

Bevestig nou dat dit die PDV bevredig

Die produkoplossing bevredig dus die PDV ongeag van watter reële skeidingskonstante gekies word

Let op: Sommige lineêre PDVs is nie skeibaar nie, bv

$$u_{xx} - u_y = x$$

Die aanname van $u(x,y) = X(x)Y(y)$ lei nie na 'n oplossing nie!