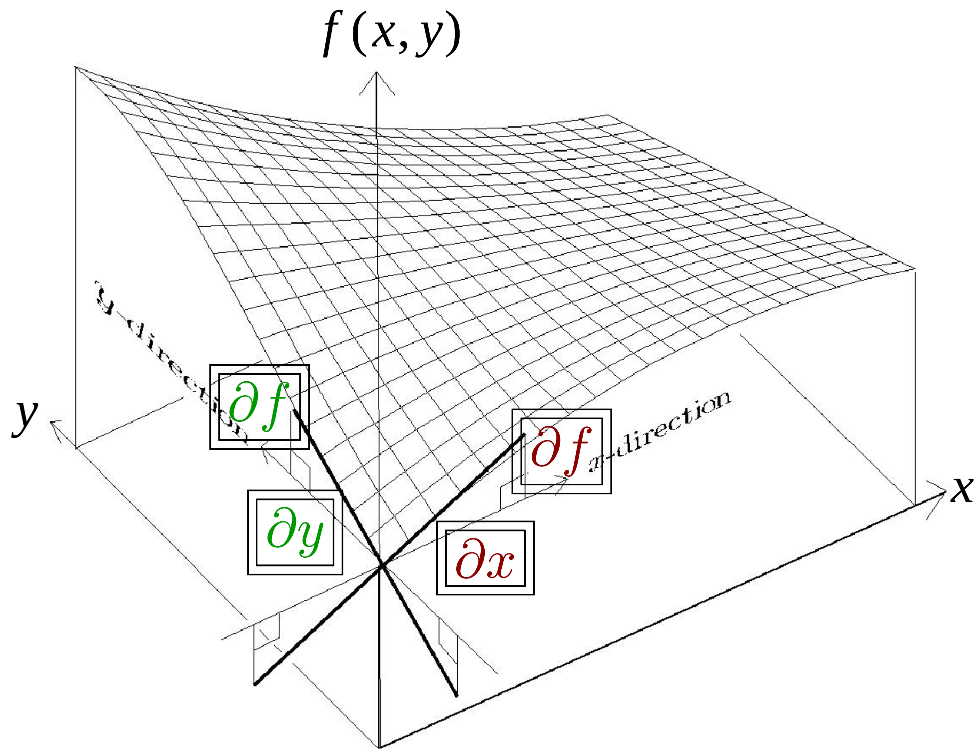
Parsiële afgeleides — Partial derivatives



Voorbeeld / *Example:* $f(x,y) = x^2y^3 + x^5\cos y$

$$\frac{\partial f}{\partial x} = 2xy^3 + 5x^4\cos y; \quad \frac{\partial f}{\partial y} = 3x^2y^2 - x^5\sin y$$

$$\frac{\partial^2 f}{\partial x^2} = 2y^3 + 20x^3\cos y; \quad \frac{\partial^2 f}{\partial x\partial y} = 6xy^2 - 5x^4\sin y$$

Hfst 13: Parsiële DVs **Chapter 13: Partial DEs**

Gewone DVs het net een onafhanklike veranderlike, en het oplossings soos/ *Ordinary DEs have one independent variable only, and have solutions like*

$$y = y(x) \quad \textbf{en}/\textit{and} \quad y = y(t)$$

Parsiële DVs het meer as 1 onafhanklike veranderlike, en het oplossings soos *Partial DEs have more than 1 independent variable only, and have solutions like*

$$y = y(x,t) \quad \textbf{en}/\textit{and} \quad u = u(x,y,z)$$

13.1: Skeibare PDVs / *Separable PDEs* (p 689)

Mees algemene lineêre 2de orde PDV / *Most general linear 2nd order PDE:*

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

Die koëffisiënte, $A = A(x,y)$, $B = B(x,y)$, $\ldots$, $G = G(x,y)$, **is almal gegee (dikwels is** A, B, $\ldots$, G **konstant)** / *The coefficients,* $A = A(x,y)$, $B = B(x,y)$, $\ldots$, $G = G(x,y)$, *are all given (often* A, B, $\ldots$, G *are constant)*

Homogeen as/*Homogeneous if* $G = 0$; **Nie-homogeen as**/*Non-homog's if* $G \neq 0$

Ons wil dus die oplossing vind/*We therefore want to find the solution* $\boxed{u = u(x,y)}$

Voorbeelde — Examples

Hitte- of diffusievergelyking (lineêr) — *Heat or diffusion equation (linear)*

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ ($c^2 \equiv$ **diffusie-konstante**/*diffusion constant*)

Golfvergelyking (lineêr) — *Wave equation (linear)*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ ($c \equiv$ **konstante (spoed)**/*constant (speed)*)

Laplace se vergelyking (lineêr) — *Laplace's equation (linear)*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Burger se vergelyking (nie-lineêr) — *Burger's equation (non-linear)*

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

Klassifikasie van PDVs / *Classification of PDEs*

Hiperbolies	as $B^2 - 4AC > 0$	*Hyperbolic*	*if* $B^2 - 4AC > 0$
Parabolies	as $B^2 - 4AC = 0$	*Parabolic*	*if* $B^2 - 4AC = 0$
Ellipties	as $B^2 - 4AC < 0$	*Elliptic*	*if* $B^2 - 4AC < 0$

Toon nou aan dat die hittevergelyking parabolies, die golfvergelyking hiperbolies en Laplace se vergelyking ellipties is / *Now show that the heat equation is parabolic, that the wave equation is hyperbolic, and that Laplace's equation is elliptic*

Produk-oplossings, skeiding van veranderlikes / *Product solutions, separation of variables*

Soek oplossings van die vorm / *Seek solutions of the form:* $\boxed{u(x,y) = X(x)Y(y)}$

$$\Rightarrow \frac{\partial u}{\partial x} = X'(x)Y(y) \qquad \frac{\partial u}{\partial y} = X(x)Y'(y)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)Y(y) \qquad \frac{\partial^2 u}{\partial y^2} = X(x)Y''(y)$$

$$\frac{\partial^2 u}{\partial x \partial y} = X'(x)Y'(y)$$

Stel in PDV $\Rightarrow$ skei PDV in 2 gewone DVs, een in X en een in Y! / *Substitute into PDE $\Rightarrow$ separate PDE into two* **ordinary** *DEs, one in X and one in Y!*

Voorbeeld 1/*Example 1:* **(p 692, Nr 4)** **Gebruik skeiding van veranderlikes en vind (indien moontlik) produkoplossings vir die PDV** / *Use separation of variables and find (if possible) product solutions for the PDE:*

$$u_x = u_y + u$$

Soek oplossings van die vorm / *Seek solutions of the form:* $\boxed{u(x,y) = X(x)Y(y)}$

$$\Rightarrow \quad u_x = X'Y \quad \text{en/and} \quad u_y = XY'$$

$$\Rightarrow \quad X'Y = XY' + XY$$

Deel met XY / *Divide by* XY*:* $\Rightarrow \quad \dfrac{X'(x)}{X(x)} = \dfrac{Y'(y)}{Y(y)} + 1$ **(1)**

Let op dat die LK van (1) net afhanklik is van x, en dat die RK van (1) net afhanklik is van y. Vgl (1) moet egter geldig wees vir alle x en y. Dit is net moontlik as die LK en RK gelyk aan 'n konstante, sê λ, is! Ons noem λ die skeidingskonstante. / *Note that the LHS of (1) is only dependant on x, and that the RHS of (1) is only dependant on y. Eqn (1) must however be valid for all x* **and** *y. This is only possible if the LHS* **and** *RHS equals a constant, say λ! We call λ the* **separation constant**.

$$\Rightarrow \quad \frac{X'}{X} = \lambda \quad \textbf{en}/\textit{and} \quad \frac{Y'}{Y} + 1 = \lambda$$

Ons het dus 1 PDV met 2 gewone DVs vervang! / *We have thus replaced 1 PDE with 2 ordinary DEs!*

Oplossing vir/*Solution for* $\frac{X'}{X} = \lambda$**:** $\quad X(x) = ce^{\lambda x}$

Oplossing vir/*Solution for* $\frac{Y'}{Y} + 1 = \lambda$**:** $\quad Y(y) = de^{(\lambda-1)y}$

Die produkoplossing word dus gegee deur / *The product solution is therefore given by*

$$u(x, y) = X(x)Y(y) = ae^{\lambda(x+y)-y},$$

vir alle konstantes/*for all constants* a **en**/*and* λ.

Bevestig!/Verify!

$$u_x = \lambda\left(ae^{\lambda(x+y)-y}\right) = \lambda u$$

$$u_y = (\lambda - 1)u = \lambda u - u$$

$$\textbf{RK}/\textit{RHS} = u_y + u = (\lambda u - u) + u = \lambda u = u_x = \textbf{LK}/\textit{LHS}$$

$\Rightarrow$ **Produkoplossing bevredig die PDV!** / *Product solution satisfies the PDE!*

Voorbeeld 2/*Example 2:* **Gebruik skeiding van veranderlikes en vind (indien moontlik) produkoplossings vir die PDV** / *Use separation of variables and find (if possible) product solutions for the PDE:*

$$u_{xx} + u_{yy} = 0$$

Soek oplossings van die vorm / *Seek solutions of the form:* $\boxed{u(x,y) = X(x)Y(y)}$

$$\Rightarrow \quad u_{xx} = X''Y \quad \textbf{en/ } \textit{and} \quad u_{yy} = XY''$$

$$\Rightarrow \quad X''Y + XY'' = 0$$

Deel met XY / *Divide by* XY*:* $\Rightarrow \dfrac{X''}{X} + \dfrac{Y''}{Y} = 0 \Rightarrow \dfrac{X''(x)}{X(x)} = -\dfrac{Y''(y)}{Y(y)}$ **(2)**

Let op dat die LK van (2) net afhanklik is van x**, en dat die RK van (2) net afhanklik is van** y**. Vgl (2) moet egter geldig wees vir alle** x **en** y**. Dit is net moontlik as die LK en RK gelyk aan 'n konstante, sê** $\pm\lambda^2$**, is! (Die kwadraat is gerieflik.)** / *Note that the LHS of (2) is only dependant on* x*, and that the RHS of (2) is only dependant on* y*. Eqn (2) must however be valid for all* x **and** y*. This is only possible if the LHS* **and** *RHS equals a constant, say* $\pm\lambda^2$*! (The "squared" is convenient.)*

Vir skeidingskonstante/*For separation constant* $\boxed{\lambda = 0}$

$$\Rightarrow \quad X'' = 0 \quad \textbf{en/}\textit{and} \quad Y'' = 0$$

$$\Rightarrow \quad X = ax + b \quad \textbf{en/}\textit{and} \quad Y = cy + d$$

$$\Rightarrow \quad u(x,y) = (ax+b)(cy+d)$$

Bevestig nou dat dit die PDV bevredig/*Now verify that this satisfies the PDE*

Vir skeidingskonstante/*For separation constant* $\boxed{+\lambda^2}$

$$\Rightarrow \quad X'' = +\lambda^2 X \quad \textbf{en/}\textit{and} \quad Y'' = -\lambda^2 Y$$

$$\Rightarrow \quad X = a\cosh(\lambda x) + b\sinh(\lambda x) \quad \textbf{en/}\textit{and}$$

$$Y = c\cos(\lambda y) + d\sin(\lambda y) \quad \textbf{dus.../}\textit{therefore...}$$

$$u(x,y) = [a\cosh(\lambda x) + b\sinh(\lambda x)]\,[c\cos(\lambda y) + d\sin(\lambda y)]$$

Bevestig nou dat dit die PDV bevredig/*Now verify that this satisfies the PDE*

Vir skeidingskonstante/*For separation constant* $\boxed{-\lambda^2}$

$$\Rightarrow \quad X'' = -\lambda^2 X \quad \textbf{en/}\textit{and} \quad Y'' = +\lambda^2 Y$$

$$\Rightarrow \quad X = a\cos(\lambda x) + b\sin(\lambda x) \quad \textbf{en/}\textit{and}$$

$$Y = c\cosh(\lambda y) + d\sinh(\lambda y) \quad \textbf{dus.../}\textit{therefore...}$$

$$u(x,y) = [a\cos(\lambda x) + b\sin(\lambda x)]\,[c\cosh(\lambda y) + d\sinh(\lambda y)]$$

Bevestig nou dat dit die PDV bevredig/*Now verify that this satisfies the PDE*

Die produkoplossing bevredig dus die PDV ongeag van watter reële skeidingskonstante gekies word / *The product solution therefore satisfies the PDE irrespective of which real separation constant is chosen*

Let op: Sommige lineêre PDVs is nie skeibaar nie, bv / *Note: Certain linear PDEs are not separable, e.g.*

$$u_{xx} - u_y = x$$

Die aanname van $u(x,y) = X(x)Y(y)$ **lei nie na 'n oplossing nie! /** *The assumption of* $u(x,y) = X(x)Y(y)$ *does not lead to a solution!*