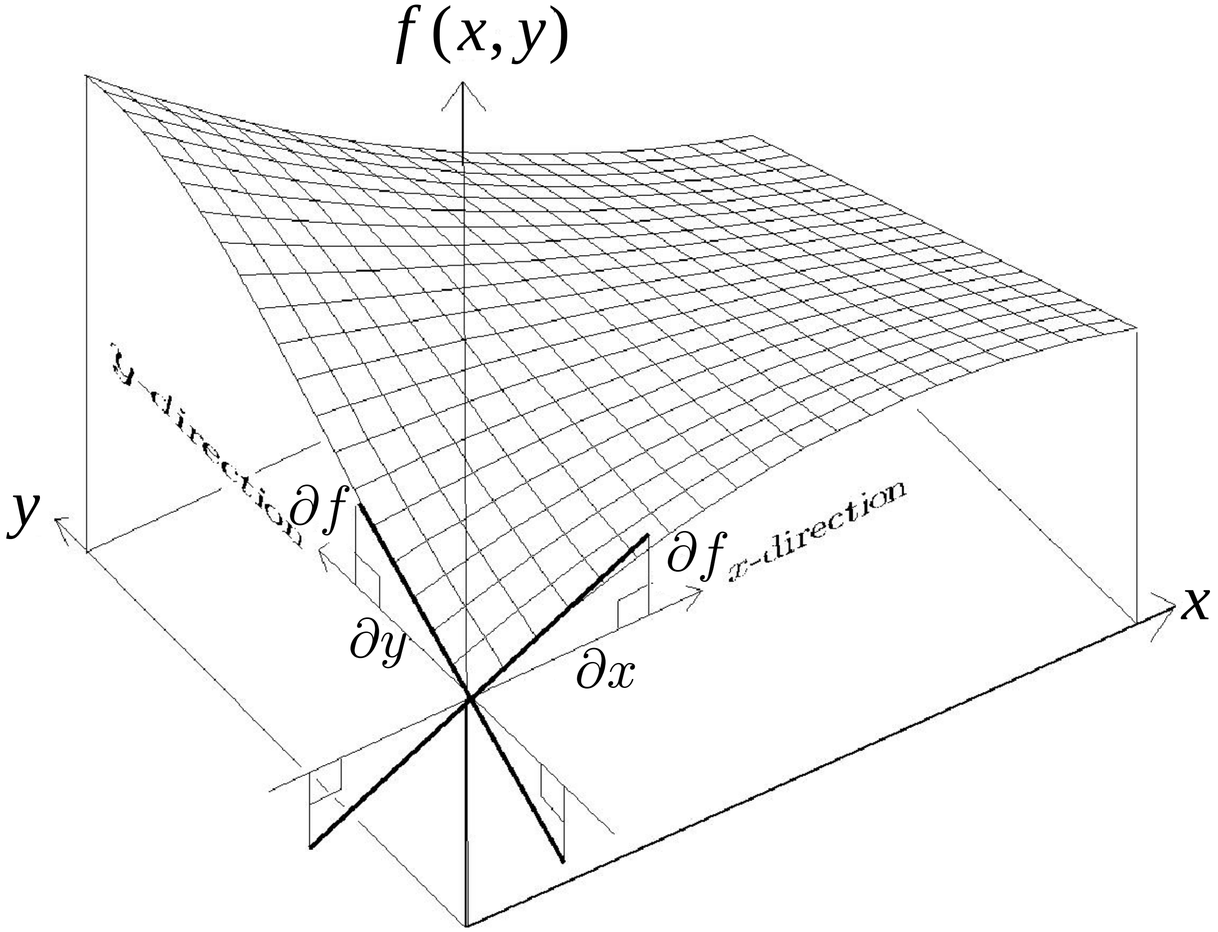
## Partial derivatives




**Example:** $f(x,y) = x^2y^3 + x^5\cos y$

$$\frac{\partial f}{\partial x} = 2xy^3 + 5x^4\cos y; \ \frac{\partial f}{\partial y} = 3x^2y^2 - x^5\sin y$$

$$\frac{\partial^2 f}{\partial x^2} = 2y^3 + 20x^3\cos y; \ \frac{\partial^2 f}{\partial x\partial y} = 6xy^2 - 5x^4\sin y$$

# Chapter 13: Partial DEs

**Ordinary DE:** Only one independent variable, e.g.

$$y = y(x) \quad \text{or} \quad y = y(t)$$

**Partial DE:** More than one independent variable, e.g.

$$y = y(x, t) \quad \text{or} \quad u = u(x, y, z)$$

## 13.1: Separable PDEs *(page 689)*

Most general linear 2nd order PDE:

$$\boxed{A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G}$$

The coefficients,

$$A = A(x, y),\ B = B(x, y),\ \ldots,\ G = G(x, y),$$

are all given (often $A$, $B$, . . . , $G$ are constant)

Homogeneous if $G = 0$; Non-homogeneous if $G \neq 0$

We therefore want to find the solution $\boxed{u = u(x, y)}$

## Examples

Heat or diffusion equation (linear)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (c^2 \equiv \text{diffusion constant})$$

Wave equation (linear) TWB224!

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (c \equiv \text{constant (speed)})$$

Laplace's equation (linear)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Burger's equation (non-linear)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

## Classification of PDEs

| | |
|---|---|
| Hyperbolic | if $B^2 - 4AC > 0$ |
| Parabolic | if $B^2 - 4AC = 0$ |
| Elliptic | if $B^2 - 4AC < 0$ |

Now show that the heat equation is parabolic, that the wave equation is hyperbolic, and that Laplace's equation is elliptic

## Product solutions, separation of variables

Seek solutions of the form: $\boxed{u(x,y) = X(x)Y(y)}$

$$\Rightarrow \frac{\partial u}{\partial x} = X'(x)Y(y) \qquad \frac{\partial u}{\partial y} = X(x)Y'(y)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)Y(y) \qquad \frac{\partial^2 u}{\partial y^2} = X(x)Y''(y)$$

$$\frac{\partial^2 u}{\partial x \partial y} = X'(x)Y'(y)$$

Substitute into PDE $\Rightarrow$ separate PDE into two **ordinary** DEs, one in $X$ and one in $Y$!

**Example 1:** *(Page 692, Nr 4)* Use separation of variables and find (if possible) product solutions for the PDE:

$$u_x = u_y + u$$

Seek solutions of the form: $\boxed{u(x,y) = X(x)Y(y)}$

$$\Rightarrow \quad u_x = X'Y \quad \text{and} \quad u_y = XY'$$

$$\Rightarrow \quad X'Y = XY' + XY$$

Divide by $XY$:

$$\Rightarrow \quad \frac{X'(x)}{X(x)} = \frac{Y'(y)}{Y(y)} + 1 \tag{1}$$

Note that the LHS of (1) is only dependant on $x$, and that the RHS of (1) is only dependant on $y$. Eqn (1) must however be valid for all $x$ **and** $y$. This is only possible if the LHS **and** RHS equals a constant, say $\lambda$! We call $\lambda$ the **separation constant**.

$$\Rightarrow \quad \frac{X'}{X} = \lambda \quad \text{and} \quad \frac{Y'}{Y} + 1 = \lambda$$

We have thus replaced 1 PDE with 2 ordinary DEs!

Solution for $\dfrac{X'}{X} = \lambda$: $\qquad X(x) = ce^{\lambda x}$

Solution for $\dfrac{Y'}{Y} + 1 = \lambda$: $Y(y) = de^{(\lambda-1)y}$

The product solution is therefore given by,

$$u(x,y) = X(x)Y(y) = ae^{\lambda(x+y)-y},$$

for all constants $a$ and $\lambda$.

**Verify!** $$u_x = \lambda\left(ae^{\lambda(x+y)-y}\right) = \lambda u$$

$$u_y = (\lambda - 1)u = \lambda u - u$$

RHS $= u_y + u = (\lambda u - u) + u = \lambda u = u_x =$ LHS

$\Rightarrow$ Product solution satisfies the PDE!

**Example 2:** Use separation of variables and find (if possible) product solutions for the PDE:

$$u_{xx} + u_{yy} = 0$$

Seek solutions of the form: $\boxed{u(x,y) = X(x)Y(y)}$

$$\Rightarrow \quad u_{xx} = X''Y \quad \text{and} \quad u_{yy} = XY''$$

$$\Rightarrow \quad X''Y + XY'' = 0$$

Divide by $XY$:

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0 \Rightarrow \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} \qquad (2)$$

Note that the LHS of (2) is only dependant on $x$, and that the RHS of (2) is only dependant on $y$. Eqn (2) must however be valid for all $x$ **and** $y$. This is only possible if the LHS **and** RHS equals a constant, say $\pm\lambda^2$! (The "squared" is convenient.)

For separation constant $\boxed{\lambda = 0}$

$$\Rightarrow \quad X'' = 0 \quad \text{and} \quad Y'' = 0$$

$$\Rightarrow \quad X = ax + b \quad \text{and} \quad Y = cy + d$$

$$\Rightarrow \quad u(x, y) = (ax + b)(cy + d)$$

Now verify that this satisfies the PDE

For separation constant $\boxed{+\lambda^2}$

$$\Rightarrow \quad X'' = +\lambda^2 X \quad \text{and} \quad Y'' = -\lambda^2 Y$$

$$\Rightarrow \quad X = a\cosh(\lambda x) + b\sinh(\lambda x) \quad \text{and}$$

$$Y = c\cos(\lambda y) + d\sin(\lambda y) \quad \text{thus...}$$

$$u(x, y) = [a\cosh(\lambda x) + b\sinh(\lambda x)]\,[c\cos(\lambda y) + d\sin(\lambda y)]$$

Now verify that this satisfies the PDE

For separation constant $\boxed{-\lambda^2}$

$$\Rightarrow \quad X'' = -\lambda^2 X \quad \text{and} \quad Y'' = +\lambda^2 Y$$

$$\Rightarrow \quad X = a\cos(\lambda x) + b\sin(\lambda x) \quad \text{and}$$

$$Y = c\cosh(\lambda y) + d\sinh(\lambda y) \quad \text{thus...}$$

$$u(x,y) = [a\cos(\lambda x) + b\sin(\lambda x)]\,[c\cosh(\lambda y) + d\sinh(\lambda y)]$$

Now verify that this satisfies the PDE

The product solution therefore satisfies the PDE irrespective of which real separation const is chosen

Note: Certain linear PDEs are not separable, e.g.

$$u_{xx} - u_y = x$$

The assumption of $u(x,y) = X(x)Y(y)$ does not lead to a solution!