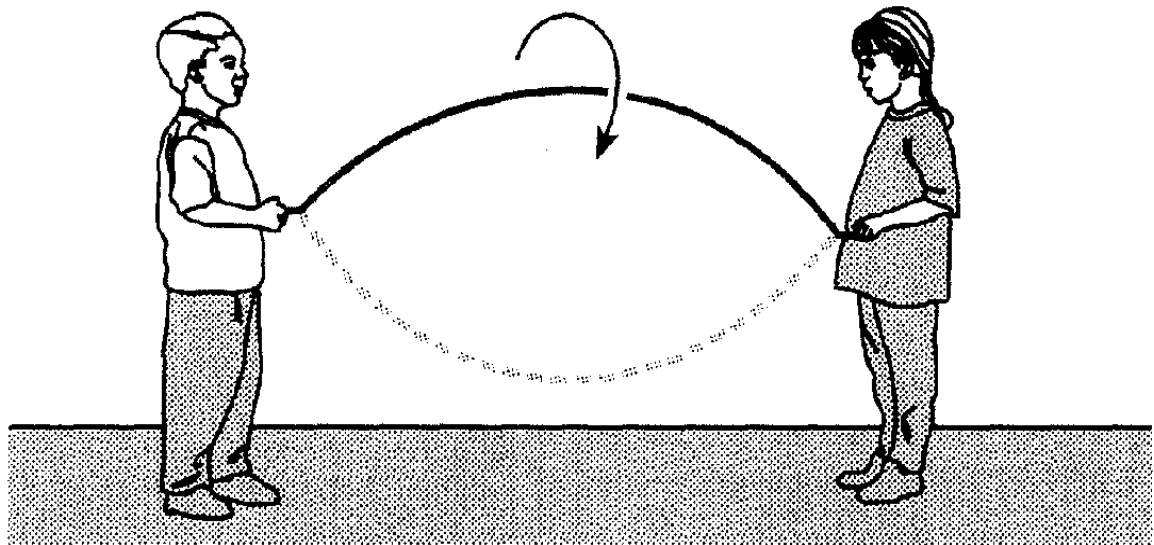


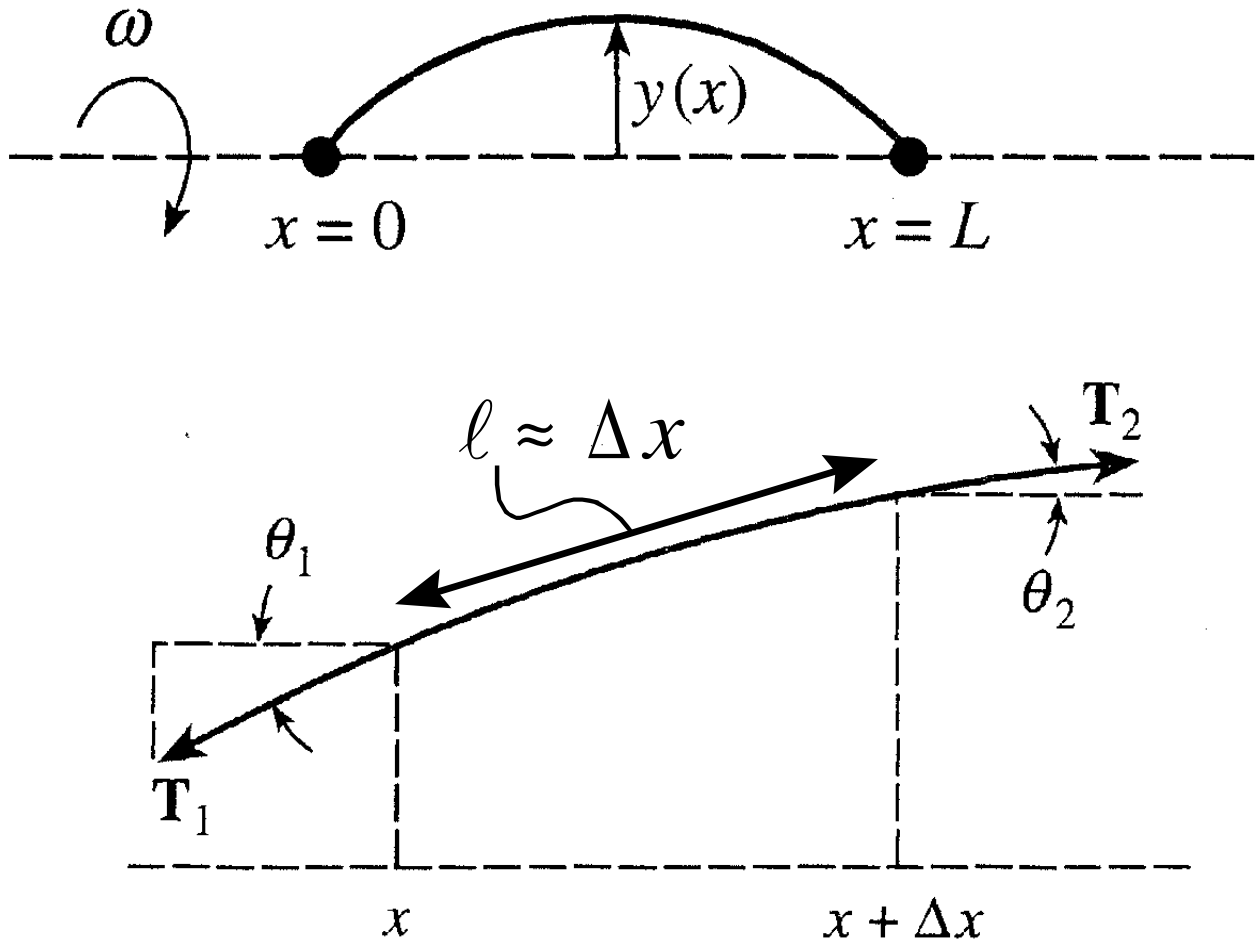
3.9 Rotating string (page 169)

Problem: For which value(s) of the angular velocity ω will the string (rope) be deflected?



Assumptions:

- (1) The angular velocity ω is constant
- (2) The tension T in the rope is constant and large – gravitation can therefore be ignored
- (3) The rope is uniform – the mass per unit length ρ is therefore constant
- (4) The deflections are small
- (5) The endpoints are fixed



Newton's second law: $(\downarrow) \Sigma F_y = ma$

$$T_1 \sin \theta_1 - T_2 \sin \theta_2 = \overbrace{(\rho \Delta x)}^m \times \overbrace{(y \omega^2)}^{a=a_n}$$

$$\text{but } T_1 = T_2 = T$$

$$\rho \Delta x y \omega^2 = T(\sin \theta_1 - \sin \theta_2)$$

but $\sin \theta_1 \approx \tan \theta_1$ and $\sin \theta_2 \approx \tan \theta_2$,
since θ_1 and θ_2 are small (deflections small)

thus $\sin \theta_1 \approx y'(x)$ and $\sin \theta_2 \approx y'(x + \Delta x)$

$$\rho y \omega^2 = \frac{T(y'(x) - y'(x + \Delta x))}{\Delta x}$$

$$\rho y \omega^2 = -T \left(\frac{y'(x + \Delta x) - y'(x)}{\Delta x} \right)$$

but $\lim_{\Delta x \rightarrow 0} \frac{y'(x + \Delta x) - y'(x)}{\Delta x} = y''(x)$

$$\text{thus } \rho y \omega^2 = -T \frac{d^2 y}{dx^2}$$

$$\Rightarrow \frac{d^2 y}{dx^2} + \frac{\rho \omega^2}{T} y = 0$$

$$\Rightarrow \frac{d^2 y}{dx^2} + \lambda y = 0 \quad \text{where} \quad \lambda = \frac{\rho \omega^2}{T}$$

with boundary cond's $y(0) = 0$ and $y(L) = 0$

(As before) Non-trivial solutions when:

$$\lambda_n = \frac{n^2 \pi^2}{L^2} = \frac{\rho \omega_n^2}{T}, \quad n = 1, 2, \dots$$

Thus critical angular velocities: $\omega = \omega_n = \frac{n\pi}{L} \sqrt{\frac{T}{\rho}}$

Non-trivial solutions: $y_n = C \sin\left(\frac{n\pi x}{L}\right)$

For $n = 1$ First critical angular velocity

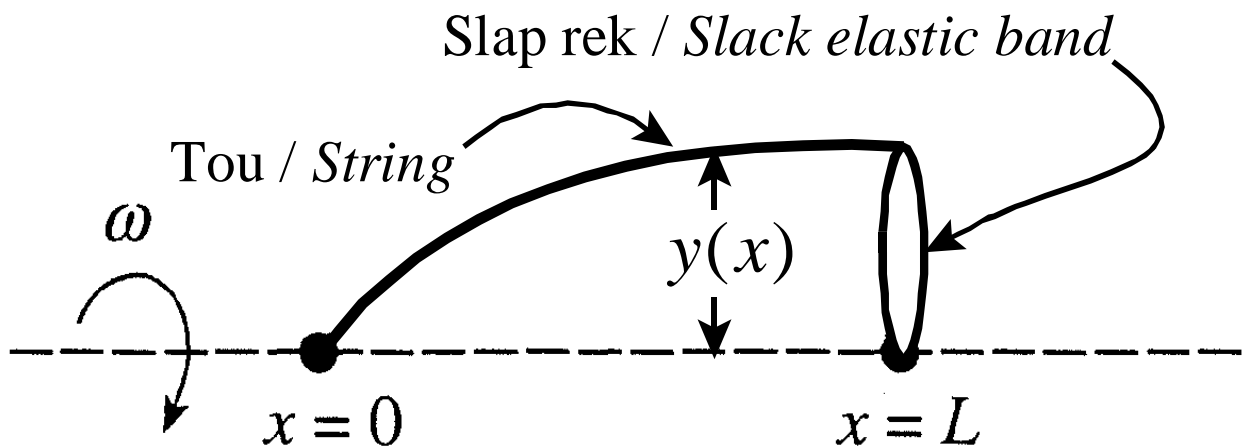
$$\omega_1 = \frac{\pi}{L} \sqrt{\frac{T}{\rho}}$$

First deflection mode: $y_1 = C \sin\left(\frac{\pi x}{L}\right)$

- $0 < \omega < \omega_1 \Rightarrow$ no deflection
- $\omega = \omega_1 \Rightarrow$ first deflection mode
- $\omega_1 < \omega < \omega_2 \Rightarrow$ no deflection
- $\omega = \omega_2 \Rightarrow$ second deflection mode

Show that the unit for ω is rad/s

Rotating string with elastic support



$$\text{DE: } \frac{d^2y}{dx^2} + \lambda y = 0 \quad \text{where} \quad \lambda = \frac{\rho\omega^2}{T}$$

with boundary cond's $y(0) = 0$ and $y'(L) = 0$

Case I: $\lambda = 0$ Show that $y \equiv 0$ (trivial sol)

Case II: $\lambda < 0$ Show that $y \equiv 0$ (trivial sol)

Case III: $\lambda > 0$ Show that...

Non-trivial solutions only when:

$$\lambda_n = \frac{(2n - 1)^2 \pi^2}{L^2} \frac{1}{4}, \quad n = 1, 2, 3, \dots$$

Non-trivial solutions (eigenfunctions):

$$y_n(x) = C \sin\left(\frac{\pi(2n - 1)}{2L} x\right)$$

$\boxed{\text{For } n = 1}$ First critical angular velocity

$$\omega_1 = \frac{\pi}{2L} \sqrt{\frac{T}{\rho}}$$

First deflection mode: $y_1 = C \sin\left(\frac{\pi x}{2L}\right)$ (sketch)
