2.8: Non-linear models *(Page 83)*

Population growth

Maltus model (linear): $\dfrac{dP}{dt} = kP,\ P(0) = P_0$

Solution: $P(t) = P_0 e^{kt}$

Note: $P \to \infty$ if $t \to \infty \Rightarrow$ not realistic!

Incorporation of births (a) and deaths (b):

$$\frac{dP}{dt} = aP - bP = \underbrace{(a-b)}_{k} P$$

Birth rate $\equiv aP$; Death rate $\equiv bP$

Still exp growth if $a > b$ & exp decay if $a < b$

Therefore not an improvement!

Verhulst's improvement: Logistic model

Improved assumption: The larger the population (P), the larger the death rate (bP), due to competition

Model: $$\frac{dP}{dt} = aP - (bP)P = aP - bP^2$$

Re-written: $$\boxed{\frac{dP}{dt} = P(a - bP),\ P(0) = P_0}$$

Note: If $P = \frac{a}{b} = K$ then $\frac{dP}{dt} = 0 \Rightarrow P$ stable

$K \equiv$ Carrying capacity of environment

Solution: Integration factors will not work, since DE is non-linear

Therefore use separation of variables and partial fractions, and show that:

$$\boxed{P(t) = \frac{\frac{a}{b}P_0}{P_0 + (\frac{a}{b} - P_0)e^{-at}}}$$

Note : If $t = 0$ then $P = P_0$

If $t \to \infty$ then $P \to \dfrac{a}{b} = K$ (carr capacity)

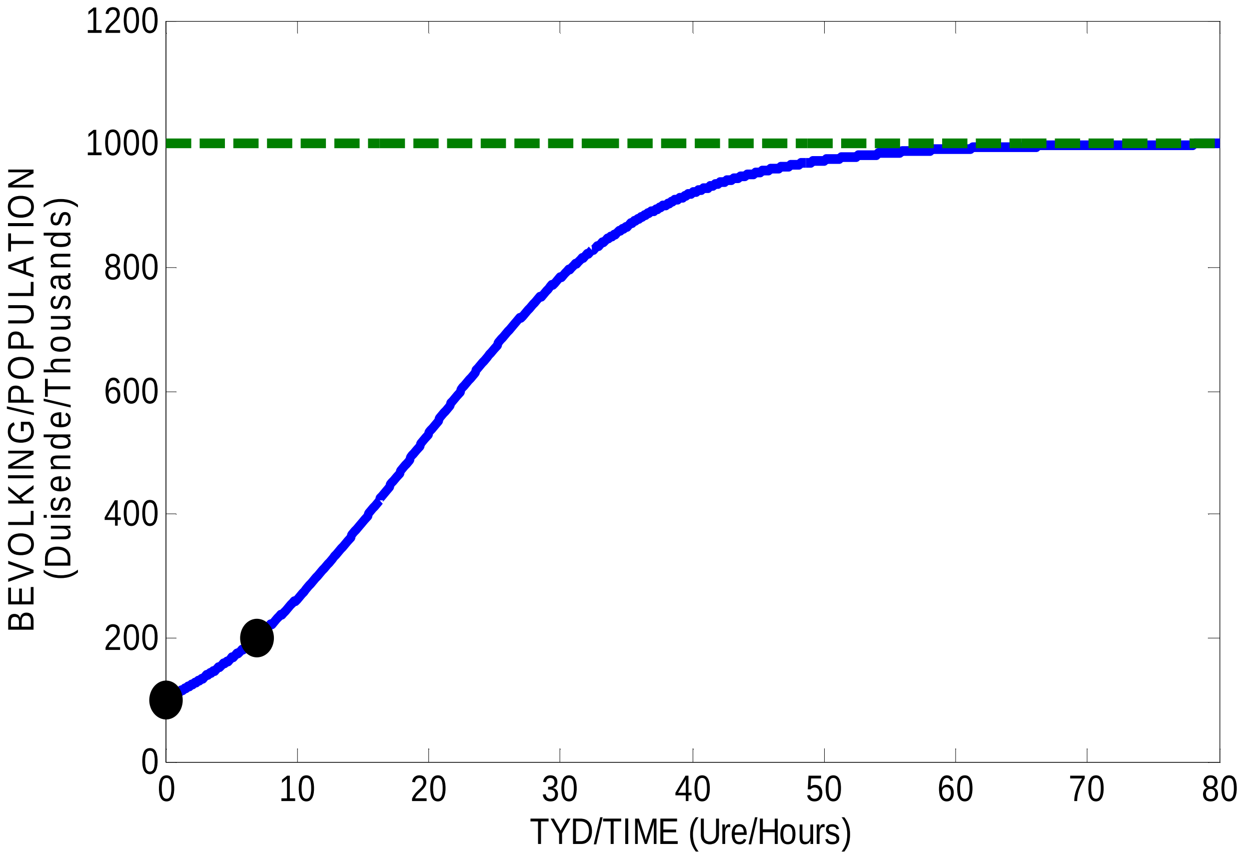
Example: The population dynamics of a colony of bacteria can be modelled by Verhulst's logistic growth model. The colony initially consists of approximately $100\,000$ individuals and approximately $200\,000$ individuals are counted 7 hours later. Eventually it appears that the population stabilizes at approximately $1\,000\,000$ individuals.

Now find an estimation for the population at any time t.

Let $P = P(t)$ be the number of bacteria (in thousands) at time t (in hours)

Answer (in thousands): $P(t) = \dfrac{1000}{1 + 9(4/9)^{t/7}}$

Logistic population growth



Graph for general case: see Fig 2.8.2 (page 84)

SELF STUDY: (page 86)
- Modifications of logistic eqn
- Chemical reactions