2.7: Linear models *(Page 72)*

Exponential growth/decay (Very important DE) :

$$\frac{dx}{dt} = kx, \text{ where } k \text{ constant}$$

Initial condition: $x(0) = x_0$

Solution methods: (a) Separation of variables
(b) Integrating factor

(a) Separation of variables

$$\int \frac{dx}{x} = k \int dt + C_1$$

$$\ln|x| = kt + C_1$$

$$|x| = e^{kt} e^{C_1} = C_2 e^{kt}$$

$$x = \pm C_2 e^{kt} = C_3 e^{kt}$$

Initial condition: $x(0) = x_0 = C_3$

Therefore: $\boxed{x(t) = x_0 e^{kt}}$ (Memorize solution!)

(b) Integrating factor, $I.F.$

$$\frac{dx}{dt} \underbrace{-k}_{P(t)} x = 0 \tag{1}$$

Multiply (1) with $I.F. = e^{\int P(t)dt} = e^{-kt}$:

$$e^{-kt}\frac{dx}{dt} - ke^{-kt}x = 0$$

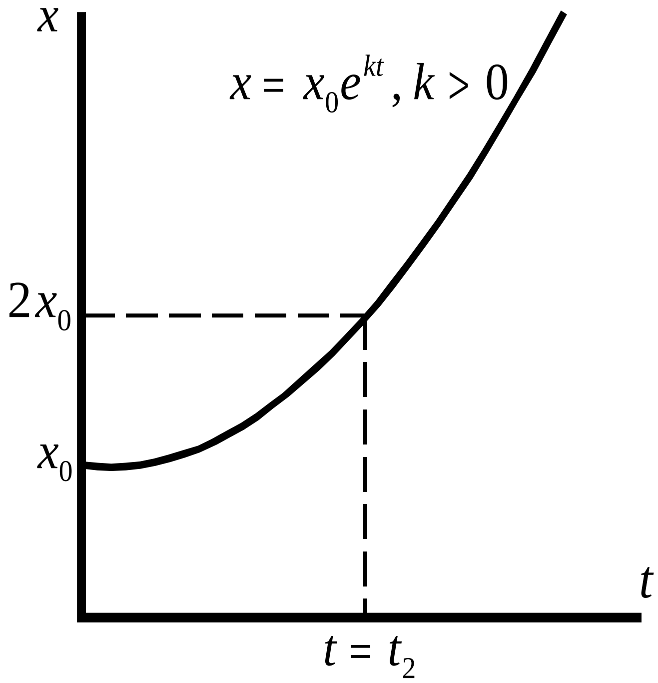
$$\frac{d}{dt}(e^{-kt}x) = 0$$

$$e^{-kt}x = C$$

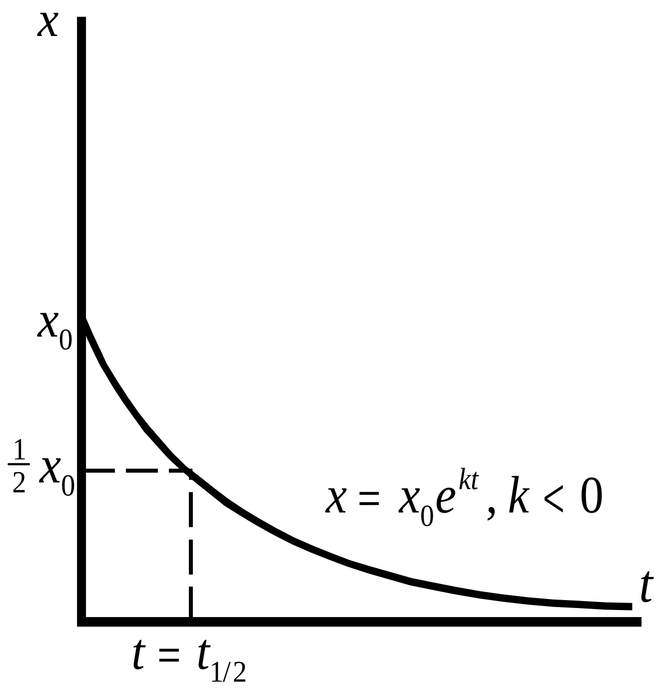
Similar to (a) we get $C = x_0$

Therefore: $\boxed{x(t) = x_0 e^{kt}}$

Exponential growth ($k > 0$): $t_2 \equiv$ doubling time



Exponential decay ($k < 0$): $t_{1/2} \equiv$ halving time



Doubling time (in the case of exponential growth)

$$x(t) = x_0 e^{kt}, \; k > 0$$

If $x = 2x_0$ then $t = t_2$:

$$\Rightarrow \quad 2x_0 = x_0 e^{kt_2}$$

$$\Rightarrow \quad \boxed{t_2 = \frac{\ln 2}{k}} \text{ and } k = \frac{\ln 2}{t_2}$$

Note: • The larger k, the smaller t_2, and vice versa
• t_2 is independent of x_0

Alternative formula for the solution:

$$\begin{aligned} x(t) &= x_0 e^{kt}, \; k > 0 \\ &= x_0 e^{\frac{t \ln 2}{t_2}} \\ &= x_0 (e^{\ln 2})^{t/t_2} \\ &= x_0 (2)^{t/t_2} \end{aligned}$$

Halving time (in the case of exponential decay)

$$x(t) = x_0 e^{kt},\ k < 0$$

If $x = \frac{1}{2}x_0$ then $t = t_{1/2}$:

$$\Rightarrow \quad \frac{1}{2}x_0 = x_0 e^{kt_{1/2}}$$

$$\Rightarrow \quad \boxed{t_{1/2} = \frac{-\ln 2}{k}} \text{ and } \ k = \frac{-\ln 2}{t_{1/2}}$$

Alternative formula for the solution:

$$\begin{aligned} x(t) &= x_0 e^{kt},\ k < 0 \\ &= x_0 e^{-\frac{t \ln 2}{t_{1/2}}} \\ &= x_0 (e^{-\ln 2})^{t/t_{1/2}} \\ &= x_0 \left(\frac{1}{2}\right)^{t/t_{1/2}} \end{aligned}$$

Applications of linear models

(1) Population growth

(2) Radioactive decay

(3) Compound interest

(4) Newton's law of cooling *(self-study)*

(5) Mixtures

(6) Electrical circuits *(self-study)*

(7) Free-fall against air resistance