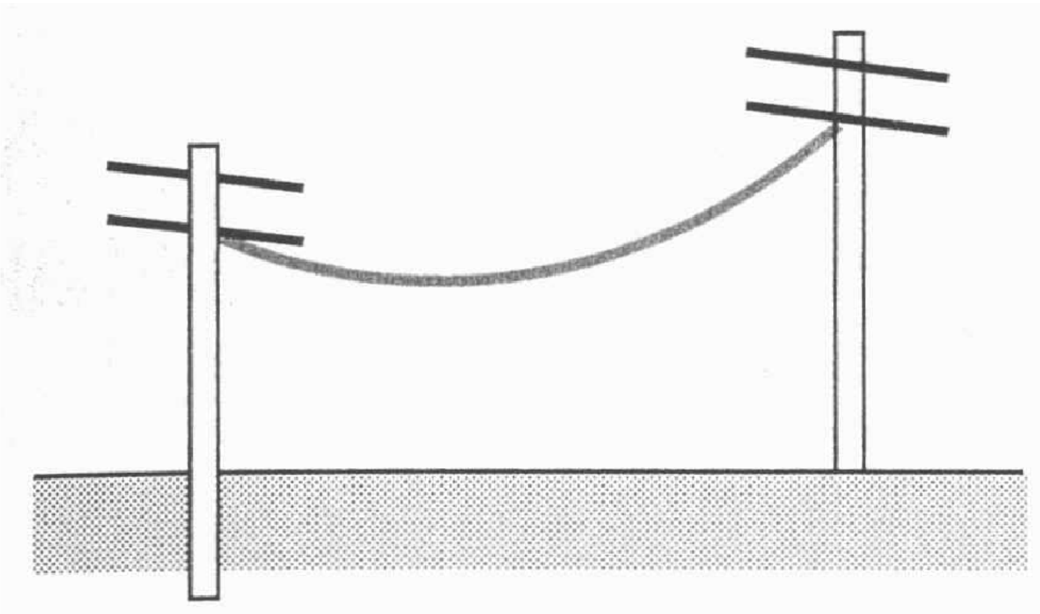
3.11 Nie-lineêre modelle

Nie-lineêre vere, stywe en pap vere, nie-lineêre pendulum (pp. 185-188): LEES

Kettinglyne *(bl 188)*

Probleem: Voorspel die vorm van 'n swaar hangende buigbare kabel, bv. 'n telefoonlyn

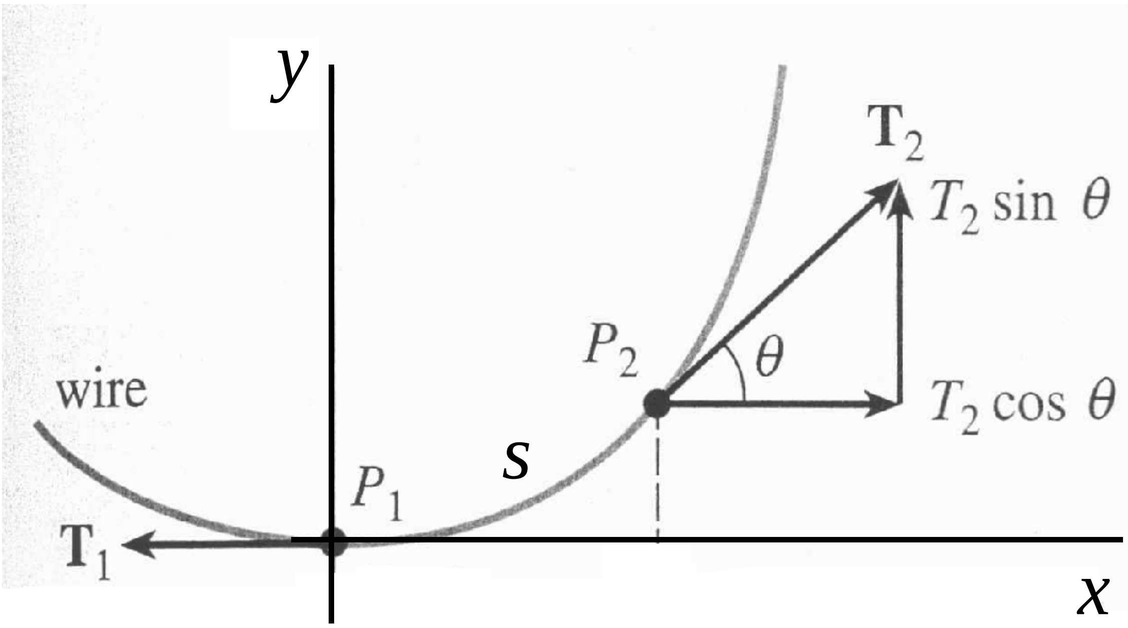


Aanname:

Die kabel is uniform met 'n konstante **gewig** per eenheidslengte ρ

Kettinglyne Tr 2

Kies die oorsprong van die assestelsel by die laagste punt van die hangende kabel/draad



(Vir ewewig) $\boxed{(\rightarrow)\,\Sigma F_x = 0}$ $\quad -T_1 + T_2\cos\theta = 0$

(Vir ewewig) $\boxed{(\uparrow)\,\Sigma F_y = 0}$ $\quad T_2\sin\theta - \rho s = 0$

Dus:

$$T_2\cos\theta = T_1 \tag{1}$$

$$T_2\sin\theta = \rho s \tag{2}$$

$$(2)/(1): \quad \Rightarrow \quad \tan\theta = \frac{\rho s}{T_1}$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{\rho s}{T_1}}$$

Neem eers aan dat die insakking **baie klein** is
$\Rightarrow s \approx x$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\rho}{T_1} x \text{ met } y(0) = 0 \text{ en } y'(0) = 0$$

Los op met skeiding van veranderlikes

$$\Rightarrow y(x) = \frac{\rho}{2T_1} x^2 \Rightarrow \textbf{Paraboliese kettinglyn}$$

Neem nou aan dat die insakking **groot** is

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\rho}{T_1} s \text{ met } y(0) = 0 \text{ en } y'(0) = 0$$

Benodig dus uitdrukking vir booglengte $s = s(x)$

$$\frac{d^2y}{dx^2} = \frac{\rho}{T_1} \frac{ds}{dx}$$

Benodig dus uitdrukking vir $\dfrac{ds}{dx}$

$$\Delta s^2 = \Delta x^2 + \Delta y^2$$

$$\Rightarrow \quad \frac{\Delta s}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \quad \text{en neem} \quad \lim_{\Delta x \to 0}$$

$$\Rightarrow \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \quad \boxed{\frac{d^2y}{dx^2} = \frac{\rho}{T_1}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} \quad \text{(nie-lineêr!)}$$

Aanvangsvoorwaardes: $y(0) = 0$ en $y'(0) = 0$

Stel $\quad u = \dfrac{dy}{dx} \quad \Rightarrow \quad \dfrac{du}{dx} = \dfrac{\rho}{T_1}\sqrt{1 + u^2}$

Los op met skeiding van veranderlikes:

$$\int \frac{du}{\sqrt{1 + u^2}} = \frac{\rho}{T_1} \int dx + C$$

$$\operatorname{arcsinh}(u) = \frac{\rho}{T_1} x + C \quad \text{(APP3, nr 17)}$$

As $x = 0$, dan $y' = 0 \Rightarrow u = 0 \Rightarrow C = 0$

$$\Rightarrow \quad u = \sinh(\frac{\rho}{T_1}x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \sinh\left(\frac{\rho}{T_1}x\right)$$

$$\Rightarrow \quad y = \frac{T_1}{\rho}\cosh\left(\frac{\rho}{T_1}x\right) + D \quad \text{(APP2, nr 20)}$$

As $x = 0$, dan $y = 0 \Rightarrow D = -\frac{T_1}{\rho}$

$$\Rightarrow \quad \boxed{y(x) = \frac{T_1}{\rho}\left(\cosh\left(\frac{\rho}{T_1}x\right) - 1\right)}$$

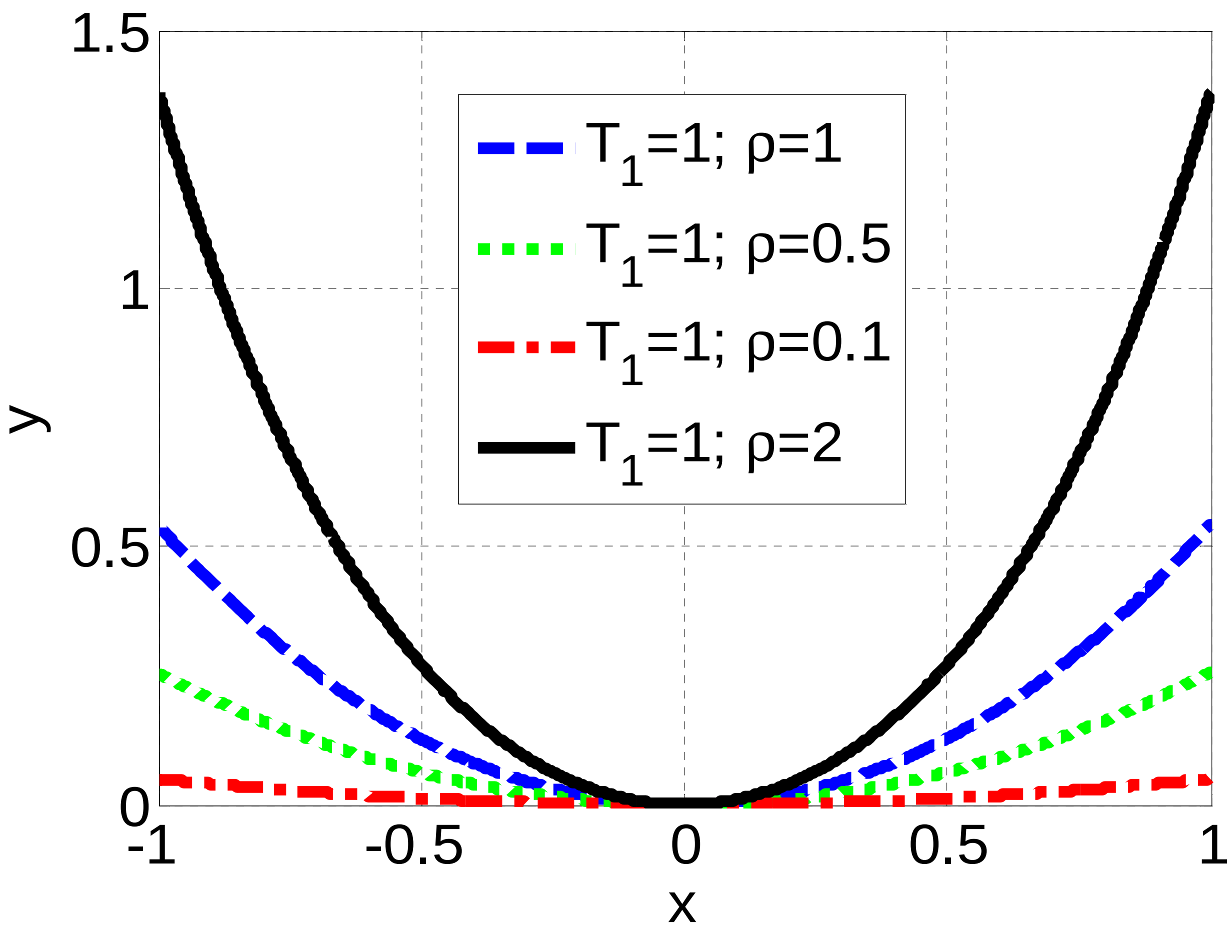
$\Rightarrow$ **Hiperboliese kettinglyn**

Toets: As $\rho \ll T_1$ (baie klein insakking)...

Taylor-reeks: $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

$$\Rightarrow \quad y(x) = \frac{T_1}{\rho}\left(1 + \frac{1}{2}\left(\frac{\rho}{T_1}x\right)^2 - 1\right)$$
$$= \frac{\rho}{2T_1}x^2 \quad \text{(Parabolies)}$$

Stem dus ooreen met resultaat vir $s \approx x$!



Addisionele formules (nie in Z&W)

Onthou:

$$T_2 \cos\theta = T_1 \quad (3)$$
$$T_2 \sin\theta = \rho s \quad (4)$$

$(3)^2 + (4)^2$: $\Rightarrow$ $T_2{}^2 = T_1{}^2 + \rho^2 s^2$

Onthou —— Formule vir die lengte van die kabel:

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

maar (soos bewys):

$$\frac{dy}{dx} = \sinh\left(\frac{\rho}{T_1}x\right)$$

$$\frac{ds}{dx} = \sqrt{1 + \sinh^2\left(\frac{\rho}{T_1}x\right)}$$

maar $\cosh^2 x - \sinh^2 x = 1$, dus

$$\frac{ds}{dx} = \cosh\left(\frac{\rho}{T_1}x\right)$$

$$s = \frac{T_1}{\rho}\sinh\left(\frac{\rho}{T_1}x\right) + C$$

As $x = 0$ dan $s = 0 \quad \Rightarrow \quad C = 0$

$$\Rightarrow \quad s = \frac{T_1}{\rho}\sinh\left(\frac{\rho}{T_1}x\right)$$

(Alternatiewelik) Onthou:

$$\frac{dy}{dx} = \frac{\rho}{T_1}s$$

$$\Rightarrow \quad s = \frac{T_1}{\rho}\frac{dy}{dx} = \frac{T_1}{\rho}\sinh\left(\frac{\rho}{T_1}x\right) \tag{5}$$

Formule vir die trekkrag in die kabel:

$$\begin{aligned} {T_2}^2 &= {T_1}^2 + \rho^2 \frac{T_1^2}{\rho^2} \sinh^2\left(\frac{\rho}{T_1}x\right) \\ &= {T_1}^2 \left(1 + \sinh^2\left(\frac{\rho}{T_1}x\right)\right) \\ &= {T_1}^2 \left(\cosh^2\left(\frac{\rho}{T_1}x\right)\right) \end{aligned}$$

$$\Rightarrow \quad T_2 = T_1 \cosh\left(\frac{\rho}{T_1}x\right)$$

maar

$$y(x) = \frac{T_1}{\rho}\left(\cosh\left(\frac{\rho}{T_1}x\right) - 1\right) \tag{6}$$

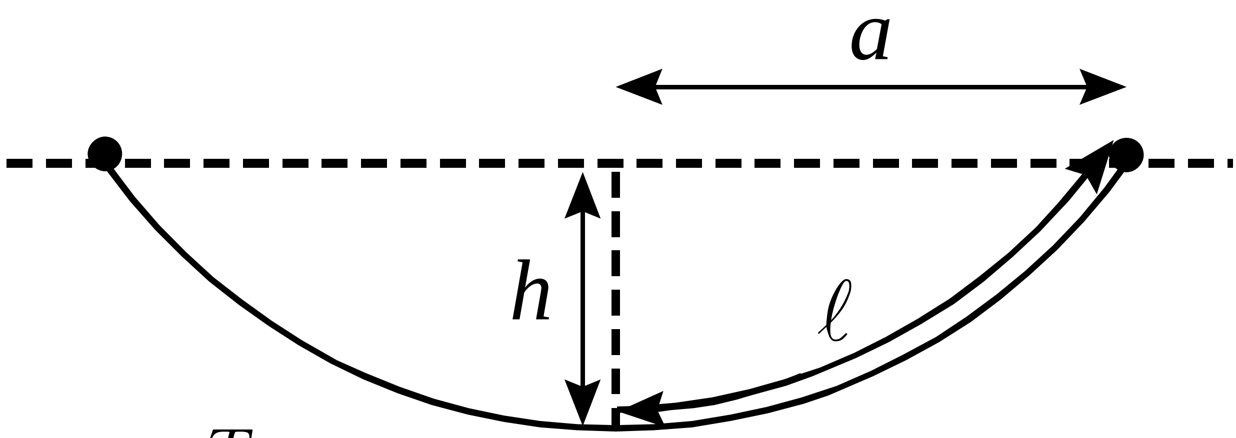
dus

$$T_2 = T_1\left(\frac{\rho}{T_1}y + 1\right)$$

dus

$$\boxed{T_2 = T_1 + \rho y}$$

Tipiese situasie: Simmetrie



Laat $c = \dfrac{T_1}{\rho}$

$a \equiv$ span; $h \equiv$ insakking; $\ell \equiv \frac{1}{2}$ lengte van die kabel

Uit (6): $$h = c[\cosh(a/c) - 1]$$

Uit (5): $$\ell = c\sinh(a/c)$$

$\Rightarrow$ 2 vergelykings met 4 onbekendes; as 2 onbekendes gegee is, kan die ander 2 dus gevind word!

<u>Probleem 1:</u> Gegee ℓ & h, vind a & c
<u>Probleem 2:</u> Gegee a & h, vind ℓ & c
<u>Probleem 3:</u> Gegee a & ℓ, vind h & c