Paragraaf 3.3 (Z&W) **(bl 112)** *Paragraph 3.3 (Z&W)* *(page 112)*

Homogene lineêre DVs met konstante koëffisiënte
Homogeneous linear DEs with constant coefficients

$$\boxed{ay'' + by' + cy = 0}$$

Voorbeeld 1/*Example 1:* $y'' + 6y' + 5y = 0$

Probeer toets-oplossing/*Try test solution,* $y = e^{mx}$, **en stel in DV**/*and substitute into DE:*

$$y = e^{mx}; \quad y' = me^{mx}; \quad y'' = m^2e^{mx}$$

$$\Rightarrow \quad m^2e^{mx} + 6me^{mx} + 5e^{mx} = 0$$

$$\Rightarrow \quad m^2 + 6m + 5 = 0$$

$$\Rightarrow (m+5)(m+1) = 0 \Rightarrow m = -1 \quad \textbf{of}/\textit{or} \quad m = -5$$

Dus twee oplossings/*Therefore two solutions:* $y = e^{-x}$ **en**/*and* $y = e^{-5x}$

Toon aan dat beide oplossings die DV bevredig!
Show that **both** *solutions satisfy the DE!*

Beginsel van superposisie: As beide y_1 en y_2 oplossings van $ay'' + by' + cy = 0$ is, dan is enige lineêre kombinasie $y = c_1y_1 + c_2y_2$ ook 'n oplossing

Principle of superposition: If both y_1 and y_2 are solutions of $ay'' + by' + cy = 0$, then any linear combination $y = c_1y_1 + c_2y_2$ is also a solution

Bewys:

Proof:

$$y = c_1y_1 + c_2y_2; \; y' = c_1y_1' + c_2y_2'; \; y'' = c_1y_1'' + c_2y_2''$$

$$\begin{aligned} ay'' + by' + cy &= a(c_1y_1'' + c_2y_2'') + b(c_1y_1' + c_2y_2') \\ &\quad + c(c_1y_1 + c_2y_2) \\ &= c_1 \underbrace{(ay_1'' + by_1' + cy_1)}_{=0} \\ &\quad + c_2 \underbrace{(ay_2'' + by_2' + cy_2)}_{=0} \\ &= 0 \end{aligned}$$

Voorbeeld 1 (vervolg): Bepaal die spesifieke oplossing van $y'' + 6y' + 5y = 0$ met die aanvangsvoorwaardes:

Example 1 (continued): Find the specific solution of $y'' + 6y' + 5y = 0$ with the initial conditions:

$$y(0) = 3; \quad y'(0) = -1$$

Algemene oplossing volgens die beginsel van superposisie:

General solution according to the principle of superposition:

$$y(x) = c_1 e^{-x} + c_2 e^{-5x}$$

$$\Rightarrow \quad y'(x) = -c_1 e^{-x} - 5c_2 e^{-5x}$$

Stel aanvangsvoorwaardes in en toon aan dat:

Substitute initial conditions and show that:

$$c_1 = \frac{7}{2}; \quad c_2 = -\frac{1}{2}$$

$$\Rightarrow \quad y(x) = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}$$

Voorbeeld 2/*Example 2:* $y'' + 4y' + 13y = 0$

Probeer toets-oplossing/*Try test solution,* $y = e^{mx}$, **en stel in DV**/*and substitute into DE:*

Hulpvergelyking/*Auxiliary equation:* $m^2 + 4m + 13 = 0 \Rightarrow m = -2 \pm 3i$

Twee oplossings/*Two solutions:* $y = e^{(-2+3i)x}$ **en**/*and* $y = e^{(-2-3i)x}$

Volgens superposisie/*Superposition:* $y(x) = e^{-2x}\left\{c_1 e^{3ix} + c_2 e^{-3ix}\right\}$

Euler se formule/*Euler's formula:*

$$e^{i\theta} = \cos\theta + i\sin\theta \quad (\theta \text{ reëel/real})$$

$$\begin{aligned} y(x) &= e^{-2x}\left[c_1(\cos 3x + i\sin 3x)\right. \\ &\quad \left. + c_2(\cos 3x - i\sin 3x)\right] \\ &= e^{-2x}\left[(c_1 + c_2)\cos 3x + i(c_1 - c_2)\sin 3x\right] \\ &= e^{-2x}\left[d_1\cos 3x + d_2\sin 3x\right] \end{aligned}$$

Voorbeeld 3/*Example 3:* $y'' - 10y' + 25y = 0$

Probeer toets-oplossing/*Try test solution,* $y = e^{mx}$, **en stel in DV**/*and substitute into DE:*

Hulpvergelyking/*Auxiliary equation:* $m^2 - 10m + 25 = 0 \Rightarrow m = 5$ (**twee keer**/*twice*)

Net een oplossing/*Only one solution:* $y_1 = e^{5x}$

Konstrueer dus tweede oplossing/*Construct second solution:* $y_2 = xe^{5x}$

Algemene oplossing/*General solution:* $y = c_1 e^{5x} + c_2 x e^{5x}$

Voorbeeld 4/*Example 4:* **(Baie belangrike DVs**/*Very important DE's:***)**

$$\boxed{\frac{d^2X}{dt^2} = +\omega^2 X}$$

Toets-oplossing/*Test solution:* $X = e^{Pt} \Rightarrow P^2 = \omega^2 \Rightarrow P = \pm\omega$

Algemene oplossing/*General solution:*
$$\begin{aligned} X &= c_1 e^{\omega t} + c_2 e^{-\omega t} \\ &= d_1 \cosh \omega t + d_2 \sinh \omega t \end{aligned}$$

(Memoriseer!/*Memorize!***)**

$$\boxed{\frac{d^2X}{dt^2} = -\omega^2 X}$$

Toets-oplossing/*Test solution:* $X = e^{Pt} \Rightarrow P^2 = -\omega^2 \Rightarrow P = \pm i\omega$

Algemene oplossing/*General solution:*
$$\begin{aligned} X &= c_1 e^{i\omega t} + c_2 e^{-i\omega t} \\ &= d_1 \cos \omega t + d_2 \sin \omega t \end{aligned}$$

(Memoriseer!/*Memorize!***)**