Paragraph 3.3 (Z&W) *(p 112)*

Homogeneous linear DEs with const coefficients

$$\boxed{ay'' + by' + cy = 0}$$

Example 1: $y'' + 6y' + 5y = 0$

Try test solution, $y = e^{mx}$, and sub into DE:

$$y = e^{mx}; \quad y' = me^{mx}; \quad y'' = m^2 e^{mx}$$

$$\Rightarrow \quad m^2 e^{mx} + 6me^{mx} + 5e^{mx} = 0$$

$$\Rightarrow \quad m^2 + 6m + 5 = 0$$

$\Rightarrow \ (m+5)(m+1) = 0 \ \Rightarrow \ m = -1$ or $m = -5$

Therefore two solutions: $y = e^{-x}$ and $y = e^{-5x}$

Show that **both** solutions satisfy the DE!

Principle of superposition: If both y_1 and y_2 are solutions of $ay'' + by' + cy = 0$, then any linear combination $y = c_1 y_1 + c_2 y_2$ is also a solution

Proof:

$$y = c_1 y_1 + c_2 y_2; \; y' = c_1 y_1' + c_2 y_2'; \; y'' = c_1 y_1'' + c_2 y_2''$$

$$\begin{aligned} ay'' + by' + cy &= a(c_1 y_1'' + c_2 y_2'') + b(c_1 y_1' + c_2 y_2') \\ &\quad + c(c_1 y_1 + c_2 y_2) \\ &= c_1 \underbrace{(a y_1'' + b y_1' + c y_1)}_{=0} \\ &\quad + c_2 \underbrace{(a y_2'' + b y_2' + c y_2)}_{=0} \\ &= 0 \end{aligned}$$

Example 1 (cont'd): Find the specific solution of $y'' + 6y' + 5y = 0$ with the initial conditions:

$$y(0) = 3; \quad y'(0) = -1$$

General solution according to the principle of superposition:

$$y(x) = c_1 e^{-x} + c_2 e^{-5x}$$

$$\Rightarrow \quad y'(x) = -c_1 e^{-x} - 5c_2 e^{-5x}$$

Substitute initial conditions and show that:

$$c_1 = \frac{7}{2}; \quad c_2 = -\frac{1}{2}$$

$$\Rightarrow \quad y(x) = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}$$

Example 2: $y'' + 4y' + 13y = 0$

Try test solution, $y = e^{mx}$, and sub into DE:

Auxiliary eqn: $m^2 + 4m + 13 = 0 \Rightarrow m = -2 \pm 3i$

Two solutions: $y = e^{(-2+3i)x}$ and $y = e^{(-2-3i)x}$

Superposition: $y = e^{-2x}\left[c_1 e^{3ix} + c_2 e^{-3ix}\right]$

Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta \quad (\theta \text{ real})$$

$$\begin{aligned} y(x) &= e^{-2x}\,[c_1(\cos 3x + i\sin 3x) \\ &\quad + c_2(\cos 3x - i\sin 3x)] \\ &= e^{-2x}\,[(c_1 + c_2)\cos 3x + i(c_1 - c_2)\sin 3x] \\ &= e^{-2x}\,[d_1\cos 3x + d_2\sin 3x] \end{aligned}$$

Example 3: $y'' - 10y' + 25y = 0$

Try test solution, $y = e^{mx}$, and sub into DE:

Aux eqn: $m^2 - 10m + 25 = 0 \Rightarrow m = 5$ (twice)

Only one solution: $y_1 = e^{5x}$

Construct second solution: $y_2 = xe^{5x}$

General solution: $y = c_1e^{5x} + c_2xe^{5x}$

Example 4 (Very important DEs):

$$\boxed{\frac{d^2X}{dt^2} = +\omega^2 X}$$

Test sol: $X = e^{Pt} \Rightarrow P^2 = \omega^2 \Rightarrow P = \pm\omega$

$$\text{Gen sol: } X = c_1 e^{\omega t} + c_2 e^{-\omega t} = d_1 \cosh \omega t + d_2 \sinh \omega t$$

(Memorize!)

$$\boxed{\frac{d^2X}{dt^2} = -\omega^2 X}$$

Test sol: $X = e^{Pt} \Rightarrow P^2 = -\omega^2 \Rightarrow P = \pm i\omega$

$$\text{Gen sol: } X = c_1 e^{i\omega t} + c_2 e^{-i\omega t} = d_1 \cos \omega t + d_2 \sin \omega t$$

(Memorize!)