Example: A thin metal rod with a length of 1 meter and a thermal diffusivity of $k = 4$ is held at a temperature of zero at its end points and is insulated from its surroundings everywhere else. Assume that the left-most point is at the origin. At time $t = 0$ it has a temperature profile of

$$30\sin(\pi x) + 10\sin(3\pi x).$$

Find the temperature profile for all times.

Mathematically formulated, the problem becomes:

$$\text{Solve} \quad u_t = 4u_{xx}, \qquad x \in [0,1], \quad t \in [0,\infty)$$

with the initial condition

$$u(x,0) = 30\sin(\pi x) + 10\sin(3\pi x) \qquad x \in [0,1]$$

and boundary conditions

$$\begin{aligned} (1): &\quad u(0,t) = 0, &\qquad t \geq 0, \\ (2): &\quad u(1,t) = 0, &\qquad t \geq 0. \end{aligned}$$

We showed in the previous lecture that infinitely many solutions $u_n(x,t),\ n = 1, 2, \ldots$ satisfy both $u_t = ku_{xx}$ and $u(0,t) = u(L,t) = 0$, where

$$u_n(x,t) = b_n e^{-k\frac{n^2\pi^2}{L^2}t}\sin(\frac{n\pi}{L}x)$$

Using the principle of superposition, any linear combination of the above solutions will also be a solution so that the general solution that satisfies both the PDE and the boundary conditions is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\frac{n^2\pi^2}{L^2}t}\sin(\frac{n\pi}{L}x)$$

With $k = 4$ and $L = 1$, the above solution becomes:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-4n^2\pi^2 t}\sin(n\pi x)$$

$\Rightarrow$

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{\infty} b_n \sin(n\pi x) \qquad (1) \\ &= b_1\sin(\pi x) + b_2\sin(2\pi x) + b_3\sin(3\pi x) + \ldots \end{aligned}$$

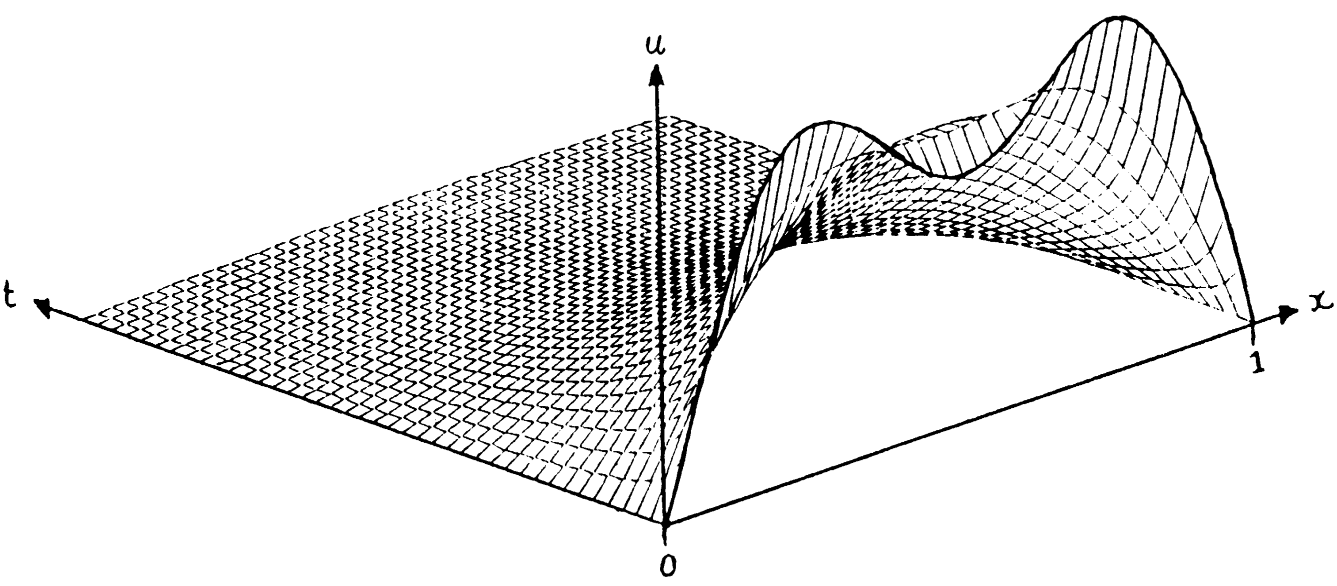
Compare (1) with the given initial condition:

$$u(x,0) = 30 \cdot \sin(\pi x) + 0 \cdot \sin(2\pi x) + 10 \cdot \sin(3\pi x)$$

$$\Rightarrow\; b_1 = 30;\; b_2 = 0;\; b_3 = 10;\; b_4 = b_5 = \ldots = 0$$

$$\begin{aligned} u(x,t) &= b_1 e^{-4(1)^2\pi^2 t}\sin(\pi x) + b_3 e^{-4(3)^2\pi^2 t}\sin(3\pi x) \\ &= 30e^{-4\pi^2 t}\sin(\pi x) + 10e^{-36\pi^2 t}\sin(3\pi x) \end{aligned}$$

The figure below shows the solution:



Example: A thin metal rod with a length of π and a thermal diffusivity of k is insulated at its end points and everywhere else. Assume that the left-most point is at the origin. At time $t = 0$ it has a temperature profile of $f(x) = \sin^2 x$. Find the temperature profile for all times.

Mathematically formulated, the problem becomes:

$$\text{Solve} \quad u_t = k u_{xx}, \quad x \in [0, \pi], \;\; t \in [0, \infty)$$

with the initial condition

$$u(x, 0) = \sin^2 x \quad x \in [0, \pi]$$

and boundary conditions

$$\begin{aligned} (1): \quad & u_x(0, t) = 0, \qquad t \geq 0, \\ (2): \quad & u_x(\pi, t) = 0, \qquad t \geq 0. \end{aligned}$$

Step 1 Seek sol's of form: $u(x,t) = X(x)T(t)$

$$\Rightarrow \quad u_t = XT' \text{ and } u_{xx} = X''T$$

$$\Rightarrow \quad XT' = kX''T$$

Now divide with kXT:

$$\Rightarrow \frac{T'}{kT} = \frac{X''}{X} = \pm\lambda^2$$

$$\Rightarrow \quad T' = \pm k\lambda^2 T \text{ and } X'' = \pm\lambda^2 X$$

Suspend step 1 for now and proceed with step 2...

Step 2 For which separation constants are the boundary conditions

$$u_x(0,t) = 0, \; u_x(\pi,t) = 0, \; t > 0$$

satisfied?

$$u(x,t) = X(x)T(t) \quad \Rightarrow u_x(x,t) = X'(x)T(t)$$

$$u_x(0,t) = X'(0)\underbrace{T(t)}_{\neq 0} = 0 \quad \Rightarrow \quad X'(0) = 0$$

$$u_x(\pi,t) = X'(\pi)\underbrace{T(t)}_{\neq 0} = 0 \quad \Rightarrow \quad X'(\pi) = 0$$

$$\Rightarrow \quad X'' = \pm\lambda^2 X, \quad X'(0) = 0, \quad X'(\pi) = 0$$

For separation constant $\boxed{\lambda = 0}$: $X(x) = ax + b$

$$\Rightarrow \quad X'(x) = a$$

$$X'(0) = a = 0 \Rightarrow \quad a = 0$$

$$X'(\pi) = a = 0 \Rightarrow \quad a = 0$$

$X(x) = b$ (constant eigenfunction)

Eigenvalue: $\lambda = 0$

For separation constants $\boxed{+\lambda^2}$

$$X(x) = a\cosh(\lambda x) + b\sinh(\lambda x)$$

$$X'(x) = \lambda a\sinh(\lambda x) + \lambda b\cosh(\lambda x)$$

$$X'(0) = \lambda b = 0 \Rightarrow \quad b = 0$$

$$X'(\pi) = \lambda a \underbrace{\sinh(\lambda\pi)}_{\neq 0} = 0 \Rightarrow \quad a = 0$$

$\Rightarrow \quad u \equiv 0 \quad \Rightarrow \quad$ Trivial solution!

Not valid – not consistent with $u(x,0) = \sin^2 x$

For separation constants $\boxed{-\lambda^2}$

$$X(x) = a\cos(\lambda x) + b\sin(\lambda x)$$

$$X'(x) = -\lambda a\sin(\lambda x) + \lambda b\cos(\lambda x)$$

$$X'(0) = \lambda b = 0 \Rightarrow \quad b = 0$$

$$X'(\pi) = -\lambda a \underbrace{\sin(\lambda\pi)}_{\text{can be 0}} = 0$$

$$\sin(\lambda\pi) = 0 \;\text{ if }\; \lambda\pi = n\pi$$

$$\Rightarrow \quad \lambda_n = n,\; n = 1, 2, \ldots \quad \text{(eigenvalues)}$$

Recall that $\lambda = 0$ is also an eigenvalue, therefore

$$\Rightarrow \quad \lambda_n = n,\; n = 0, 1, 2, \ldots$$

$$\Rightarrow \quad X_n(x) = a_n \cos(\lambda_n x) \quad \text{(eigenfunctions)}$$

$$\Rightarrow \quad T_n' = -k{\lambda_n}^2 T \quad \Rightarrow \quad T_n(t) = c_n e^{-k{\lambda_n}^2 t}$$

$$\Rightarrow \quad u_n(x,t) = A_n e^{-k{\lambda_n}^2 t} \cos(\lambda_n x)$$

Step 3 Using the principle of superposition, any linear combination of the above solutions will also be a solution so that the general solution that satisfies both the PDE and the boundary conditions is given by

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} A_n e^{-k\lambda_n{}^2 t} \cos(\lambda_n x) \\ &= \sum_{n=0}^{\infty} A_n e^{-kn^2 t} \cos(nx) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad u(x,0) &= \sum_{n=0}^{\infty} A_n \cos(nx) \\ &= A_0 + A_1 \cos(x) + A_2 \cos(2x) + \ldots \end{aligned}$$

Compare this with the given initial condition:

$$\begin{aligned} u(x,0) &= \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) \\ &= \frac{1}{2} + 0 \cdot \cos(x) - \frac{1}{2}\cos(2x) + 0 \cdot \cos(3x) + . \end{aligned}$$

$$\Rightarrow \; A_0 = \frac{1}{2};\; A_1 = 0;\; A_2 = -\frac{1}{2};\; A_3 = A_4 = \ldots = 0$$

$$\Rightarrow \; u(x,t) = \frac{1}{2} - \frac{1}{2} e^{-4kt} \cos(2x)$$