Die hittevergelyking II / The heat equation II

Beskou die spesifieke probleem: / *Consider the specific problem:*

$$u_t = k u_{xx}, \quad k > 0$$

Aanvangsvoorwaarde: / *Initial condition:*

$$u(x, 0) = f(x),\ 0 < x < L$$

Randvoorwaardes: / *Boundary conditions*

$$u(0, t) = 0,\ t > 0$$

$$u(L, t) = 0,\ t > 0$$

Oplossingsmetodiek / *Solution methodology* (p 697)

1. **Bepaal alle produkoplossings**
2. **Kies daardie oplossings in stap 1 wat die randvoorwaardes bevredig**
3. **Probeer die aanvangsvoorwaarde(s) bevredig deur 'n lineêre kombinasie van die oplossings in stap 2 te neem**

1. *Determine all product solutions*
2. *Choose those solutions in* **step 1** *that satisfy the boundary conditions*
3. *Try to satisfy the initial condition(s) by taking a linear combination of the solutions in* **step 2**

Die beginsel van superposisie / Die principle of superposition (p 690)

As $v(x,t)$ en $w(x,t)$ die hittevergelyking bevredig, bevredig $z(x,t) = c\,v(x,t) + d\,w(x,t)$ ook die hittevergelyking / **When $v(x,t)$ and $w(x,t)$ satisfy the heat equation, then $z(x,t) = c\,v(x,t) + d\,w(x,t)$ also satisfies the heat equation**

Bewys/Proof:

$$v_t = kv_{xx} \quad \text{en/and} \quad w_t = kw_{xx}$$

$$\begin{aligned} \Rightarrow \quad z_t &= cv_t + dw_t \\ &= c(kv_{xx}) + d(kw_{xx}) \\ &= k(cv_{xx} + dw_{xx}) \\ &= kz_{xx} \end{aligned}$$

$\Rightarrow \quad z(x,t)$ is ook 'n oplossing!/*is also a solution!*

Stap 1/Step 1 Soek opl's van vorm / *Seek sol's of form:* $u(x,t) = X(x)T(t)$

$$\Rightarrow \quad u_t = XT' \quad \text{en/}\textit{and} \quad u_{xx} = X''T$$

$$\Rightarrow \quad XT' = kX''T$$

Deel nou met kXT: *Now divide with kXT:*

$$\Rightarrow \frac{T'}{kT} = \frac{X''}{X} = \pm\lambda^2$$

$$\Rightarrow \quad T' = \pm k\lambda^2 T \quad \text{en/}\textit{and} \quad X'' = \pm\lambda^2 X$$

Vir skeidingskonstante / *For separation constant* $\boxed{\lambda = 0}$

$$\Rightarrow \quad X'' = 0 \quad \textbf{en}//\textit{and} \quad T' = 0$$

$$\Rightarrow \quad X = cx + d \quad \textbf{en}/\textit{and} \quad T = e$$

$$\Rightarrow \quad u(x,t) = X(x)T(t) = ax + b$$

Bevestig nou dat dit die PDV bevredig / *Now verify that this satisfies the PDE*

Vir skeidingskonstantes / *For separation constants* $\boxed{+\lambda^2}$

$$\Rightarrow \quad X'' = +\lambda^2 X \quad \textbf{en}/\textit{and} \quad T' = +k\lambda^2 T$$

$$\Rightarrow \quad X = a\cosh(\lambda x) + b\sinh(\lambda x) \quad \textbf{en}/\textit{and}$$

$$T = ce^{k\lambda^2 t} \quad \textbf{dus...}/\textit{therefore...}$$

$$u(x,t) = e^{k\lambda^2 t}\left[a\cosh(\lambda x) + b\sinh(\lambda x)\right]$$

Bevestig nou dat dit die PDV bevredig / *Now verify that this satisfies the PDE*

Vir skeidingskonstantes / *For separation constants* $\boxed{-\lambda^2}$

$$\Rightarrow \quad X'' = -\lambda^2 X \quad \textbf{en}/\textit{and} \quad T' = -k\lambda^2 T$$

$$\Rightarrow \quad X = a\cos(\lambda x) + b\sin(\lambda x) \quad \textbf{en}/\textit{and}$$

$$T = ce^{-k\lambda^2 t} \quad \textbf{dus...}/\textit{therefore...}$$

$$u(x,t) = e^{-k\lambda^2 t}\left[a\cos(\lambda x) + b\sin(\lambda x)\right]$$

Bevestig nou dat dit die PDV bevredig / *Now verify that this satisfies the PDE*

Stap 2/Step 2 **Vir watter skeidingskonstantes word die randvoorwaardes/** *For which separation constants are the boundary conditions*

$$u(0,t) = 0,\ u(L,t) = 0,\ t > 0$$

bevredig/*satisfied***?**

Vir skeidingskonstante / *For separation constant* $\boxed{\lambda = 0}$**:** $u(x,t) = ax + b$

As/*When* $x = 0$ **dan/***then* $u = 0 \Rightarrow b = 0$

As/*When* $x = L$ **dan/***then* $u = 0 \Rightarrow aL = 0 \Rightarrow a = 0$

$$\Rightarrow \quad u \equiv 0 \quad \Rightarrow \quad \textbf{Triviale oplossing!}/\textit{Trivial solution!}$$

Vir skeidingskonstantes / *For separation constants* $\boxed{+\lambda^2}$

$$u(x,t) = e^{k\lambda^2 t}\left[a\cosh(\lambda x) + b\sinh(\lambda x)\right]$$

As/*When* $x = 0$ **dan**/*then* $u = 0 \Rightarrow \underbrace{e^{k\lambda^2 t}}_{\boxed{\neq 0}} \cdot a = 0 \Rightarrow a = 0$

As/*When* $x = L$ **dan**/*then* $u = 0 \Rightarrow \underbrace{e^{k\lambda^2 t}}_{\boxed{\neq 0}} \cdot b \cdot \underbrace{\sinh(\lambda L)}_{\boxed{> 0}} = 0$

$\Rightarrow \quad b = 0 \quad \Rightarrow \quad u \equiv 0 \quad \Rightarrow$ **Triviale oplossing!**/*Trivial solution!*

Vir skeidingskonstantes / *For separation constants* $\boxed{-\lambda^2}$

$$u(x,t) = e^{-k\lambda^2 t}\left[a\cos(\lambda x) + b\sin(\lambda x)\right]$$

As/*When* $x = 0$ **dan**/*then* $u = 0 \Rightarrow \underbrace{e^{-k\lambda^2 t}}_{\boxed{\neq 0}} \cdot a = 0 \Rightarrow a = 0$

As/*When* $x = L$ **dan**/*then* $u = 0 \Rightarrow \underbrace{e^{-k\lambda^2 t}}_{\boxed{\neq 0}} \cdot b \cdot \underbrace{\sin(\lambda L)}_{\textbf{can be } 0} = 0$

$\sin(\lambda L) = 0$ **as**/*if* $\lambda L = n\pi, \; n = 1, 2, 3, \ldots$

$$\Rightarrow \quad \lambda_n = \frac{n\pi}{L}, \; n = 1, 2, 3, \ldots$$

$\Rightarrow$ **Oneindig baie oplossings/***Infinitely many solutions:*

$$u_n = b_n e^{-k\frac{n^2\pi^2}{L^2}t} \sin(\frac{n\pi}{L}x) = u_n(x,t)$$

Hierdie oplossings bevredig die PDV en die randvoorwaardes! / *These solutions satisfy the PDE* **and** *the boundary conditions!*

Stap 3/Step 3 **Probeer die aanvangsvoorwaarde(s) bevredig deur 'n lineêre kombinasie van die oplossings in** Stap 2/Step 2 **te neem** / *Try to satisfy the initial condition(s) by taking a linear combination of the solutions in* Stap 2/Step 2

Volgens die beginsel van superposisie is enige lineêre kombinasie van die oplossings ook 'n oplossing, dus... / *According to principle of superposition, any linear combination of solutions is also a solution, therefore...*

Mees algemene oplossing / *Most general solution:*

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\frac{n^2\pi^2}{L^2}t} \sin(\frac{n\pi}{L}x)$$

As/*When* $t = 0$ **dan/***then* $u = f(x)$

$$\Rightarrow \quad f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x)$$

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As $f(x)$ geskryf kan word as 'n lineêre kombinasie van die sinus-terme,
If $f(x)$ can be written as a linear combination of the sine terms,

$$\sin(\frac{n\pi}{L}x),$$

kan die koëffisiënte b_n bepaal word deur koëffisiënte links en regs te vergelyk!
then the coefficients b_n can be found by comparing coefficients on the LHS and RHS!

Voorbeeld: *Example:*

$$u_t = u_{xx}$$

Aanvangsvoorwaarde: *Initial condition:*

$$u(x,0) = \sin x,\ 0 < x < \pi$$

Randvoorwaardes: *Boundary conditions*

$$u(0,t) = 0,\ t > 0$$

$$u(\pi,t) = 0,\ t > 0$$

Mees algemene oplossing: *Most general solution:*

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx)$$

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As/ *When* $t = 0$ **dan/** *then* $u = \sin x$

$$\Rightarrow \quad \sin x = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Vergelyk koëffisiënte links en regs / *Compare coefficients on the LHS and RHS:*

$$1 \cdot \sin(x) + 0 \cdot \sin(2x) + 0 \cdot \sin(3x) + \dots$$

$$= b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x) + \dots$$

$$\Rightarrow \quad b_1 = 1, \quad b_2 = b_3 = \dots = 0$$

$$\Rightarrow \quad u(x,t) = e^{-t} \sin x$$

Let op: $u \to 0$ **(gestadigde toestand) as** $t \to \infty$
Note: $u \to 0$ *(steady state) when* $t \to \infty$

Teken nou die oplossing op een assestelsel / *Now sketch the solution on one system of axes*