The heat equation II

Consider the specific problem:

$$u_t = ku_{xx}, \quad k > 0$$

Initial condition:

$$u(x,0) = f(x),\ 0 < x < L$$

Boundary conditions:

$$u(0,t) = 0,\ t > 0$$

$$u(L,t) = 0,\ t > 0$$

Solution methodology *(page 697)*

Step 1 Determine all product solutions

Step 2 Choose those solutions in Step 1 that satisfy the boundary conditions

Step 3 Try to satisfy the initial condition(s) by taking a linear combination of the solutions in Step 2

Die principle of superposition *(page 690)*

When $v(x,t)$ and $w(x,t)$ satisfy the heat equation, then $z(x,t) = c\,v(x,t) + d\,w(x,t)$ also satisfies the heat equation

Proof: $v_t = kv_{xx}$ and $w_t = kw_{xx}$

$$\begin{aligned} \Rightarrow \quad z_t &= cv_t + dw_t \\ &= c(kv_{xx}) + d(kw_{xx}) \\ &= k(cv_{xx} + dw_{xx}) \\ &= kz_{xx} \end{aligned}$$

$\Rightarrow \quad z(x,t)$ is also a solution!

Step 1 Seek sol's of form: $u(x,t) = X(x)T(t)$

$$\Rightarrow \quad u_t = XT' \text{ and } u_{xx} = X''T$$

$$\Rightarrow \quad XT' = kX''T$$

Now divide with kXT:

$$\Rightarrow \frac{T'}{kT} = \frac{X''}{X} = \pm\lambda^2$$

$$\Rightarrow \quad T' = \pm k\lambda^2 T \text{ and } X'' = \pm\lambda^2 X$$

For separation constant $\boxed{\lambda = 0}$

$$\Rightarrow \quad X'' = 0 \quad \text{and} \quad T' = 0$$

$$\Rightarrow \quad X = cx + d \quad \text{and} \quad T = e$$

$$\Rightarrow \quad u(x,t) = X(x)T(t) = ax + b$$

Now verify that this satisfies the PDE

For separation constants $\boxed{+\lambda^2}$

$$\Rightarrow \quad X'' = +\lambda^2 X \quad \text{and} \quad T' = +k\lambda^2 T$$

$$\Rightarrow \quad X = a\cosh(\lambda x) + b\sinh(\lambda x) \quad \text{and}$$

$$T = ce^{k\lambda^2 t} \quad \text{thus...}$$

$$u(x,t) = e^{k\lambda^2 t}\left[a\cosh(\lambda x) + b\sinh(\lambda x)\right]$$

Now verify that this satisfies the PDE

For separation constants $\boxed{-\lambda^2}$

$$\Rightarrow \quad X'' = -\lambda^2 X \quad \text{and} \quad T' = -k\lambda^2 T$$

$$\Rightarrow \quad X = a\cos(\lambda x) + b\sin(\lambda x) \quad \text{and}$$

$$T = ce^{-k\lambda^2 t} \quad \text{thus...}$$

$$u(x,t) = e^{-k\lambda^2 t}\left[a\cos(\lambda x) + b\sin(\lambda x)\right]$$

Now verify that this satisfies the PDE

$\boxed{\text{Step 2}}$ For which separation constants are the boundary conditions

$$u(0,t) = 0,\ u(L,t) = 0,\ t > 0$$

satisfied?

For separation constant $\boxed{\lambda = 0}$: $u(x,t) = ax + b$

When $x = 0$ then $u = 0 \Rightarrow b = 0$

When $x = L$ then $u = 0 \Rightarrow aL = 0 \Rightarrow a = 0$

$$\Rightarrow \quad u \equiv 0 \quad \Rightarrow \quad \text{Trivial solution!}$$

For separation constants $\boxed{+\lambda^2}$

$$u(x,t) = e^{k\lambda^2 t}\left[a\cosh(\lambda x) + b\sinh(\lambda x)\right]$$

When $x = 0$ then $u = 0 \Rightarrow \underbrace{e^{k\lambda^2 t}}_{\neq 0} \cdot a = 0 \Rightarrow a = 0$

When $x = L$ then $u = 0 \Rightarrow \underbrace{e^{k\lambda^2 t}}_{\neq 0} \cdot b \cdot \underbrace{\sinh(\lambda L)}_{>0} = 0$

$\Rightarrow \quad b = 0 \quad \Rightarrow \quad u \equiv 0 \quad \Rightarrow$ Trivial solution!

For separation constants $\boxed{-\lambda^2}$

$$u(x,t) = e^{-k\lambda^2 t}\left[a\cos(\lambda x) + b\sin(\lambda x)\right]$$

When $x = 0$ then $u = 0 \Rightarrow \underbrace{e^{-k\lambda^2 t}}_{\neq 0} \cdot a = 0 \Rightarrow a = 0$

When $x = L$ then $u = 0 \Rightarrow \underbrace{e^{-k\lambda^2 t}}_{\neq 0} \cdot b \cdot \underbrace{\sin(\lambda L)}_{\text{can be } 0} = 0$

$\sin(\lambda L) = 0$ when $\lambda L = n\pi,\ n = 1, 2, 3, \ldots$

$$\Rightarrow \quad \lambda_n = \frac{n\pi}{L},\ n = 1, 2, 3, \ldots$$

$$\Rightarrow \quad \text{Infinitely many solutions:}$$

$$u_n = b_n e^{-k\frac{n^2\pi^2}{L^2}t} \sin(\frac{n\pi}{L}x) = u_n(x,t)$$

These solutions satisfy the PDE **and** the boundary conditions!

Step 3 Try to satisfy the initial condition(s) by taking a linear combination of the solutions in Step 2

According to principle of superposition, any linear combination of solutions is also a solution, thus...

Most general solution:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\frac{n^2\pi^2}{L^2}t} \sin(\frac{n\pi}{L}x)$$

When $t = 0$ then $u = f(x)$

$$\Rightarrow \quad f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x)$$

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If $f(x)$ can be written as a linear combination of the sine terms,

$$\sin(\frac{n\pi}{L}x),$$

then the coefficients b_n can be determined by comparing coefficients on the LHS and RHS!

Example: $$u_t = u_{xx}$$

Initial condition:

$$u(x,0) = \sin x,\ 0 < x < \pi$$

Boundary conditions:

$$u(0,t) = 0,\ t > 0$$

$$u(\pi,t) = 0,\ t > 0$$

Most general solution:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin(nx)$$

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When $t = 0$ then $u = \sin x$

$$\Rightarrow \quad \sin x = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Compare coefficients on the LHS and RHS:

$$1 \cdot \sin(x) + 0 \cdot \sin(2x) + 0 \cdot \sin(3x) + \dots$$
$$= b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x) + \dots$$

$$\Rightarrow \quad b_1 = 1, \quad b_2 = b_3 = \dots = 0$$

$$\Rightarrow \quad u(x,t) = e^{-t} \sin x$$

Note: $u \to 0$ (steady state) when $t \to \infty$

Now sketch the solution on one system of axes