2.8: Free-fall against air resistance II

(like prob 17, p 89)

Person falls (from rest) from airplane...

Let downward direction be positive
(displacement $\equiv y(t)$; velocity $\equiv v(t)$)

From rest: $y(0) = 0$; $v(0) = 0$

<u>Without air resistance</u> ($R = 0$)

$$m\frac{dv}{dt} = mg \quad \Rightarrow \quad \boxed{v(t) = gt}$$

<u>With air resistance</u> (Linear model: $R \propto v$)

$$m\frac{dv}{dt} = mg - kv \quad \Rightarrow \quad \boxed{v(t) = \frac{mg}{k}\left(1 - e^{-kt/m}\right)}$$

Limit velocity is $\dfrac{mg}{k}$, where $[k] = \text{kg/s}$

<u>With air resistance</u> (Non-linear model: $R \propto v^2$)

$$m\frac{dv}{dt} = mg - kv^2 \quad \Rightarrow \quad \boxed{v(t) = ?}$$

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Unit for k: $[k] = \text{kg/m}$

Re-write DE: $\dfrac{dv}{dt} = -\dfrac{k}{m}(v^2 - a^2)$, where $a^2 = \dfrac{mg}{k}$

Separation of variables:

$$\frac{1}{2a}\left(\ln|v-a| - \ln|v+a|\right) = -\frac{k}{m}t + C$$

Initial condition: $v(0) = 0 \Rightarrow C = 0$

$$\ln\left|\frac{v-a}{v+a}\right| = -\frac{2ak}{m}t$$

$$\frac{ak}{m} = \sqrt{\frac{gk}{m}}$$

$$\left|\frac{v-a}{v+a}\right| = e^{-2\sqrt{\frac{gk}{m}}t}$$

$$\frac{v-a}{v+a} = \pm e^{-2\sqrt{\frac{gk}{m}}t}$$

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When $t = 0$ and $\boxed{+}$ is chosen, then $a = 0 \Rightarrow m = 0$ $\Rightarrow$ contradiction

When $t = 0$ and $\boxed{-}$ is chosen, then $v = 0 \Rightarrow$ satisfies initial condition

Thus choose $\boxed{-}$, so that $\dfrac{v-a}{v+a} = -e^{-2\sqrt{\frac{gk}{m}}t}$

Make v the subject of the equation:

$$v(t) = a\left(\frac{1 - e^{-2\sqrt{\frac{gk}{m}}t}}{1 + e^{-2\sqrt{\frac{gk}{m}}t}}\right)$$

Note: $v \to a$ as $t \to \infty$

$\Rightarrow$ Limit velocity: $a = \sqrt{\dfrac{mg}{k}}$ (verify unit)

Multiply with $e^{\sqrt{\frac{gk}{m}}t}$ above and below:

$$v(t) = a\left(\frac{e^{\sqrt{\frac{gk}{m}}t} - e^{-\sqrt{\frac{gk}{m}}t}}{e^{\sqrt{\frac{gk}{m}}t} + e^{-\sqrt{\frac{gk}{m}}t}}\right) = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}} \cdot t\right)$$

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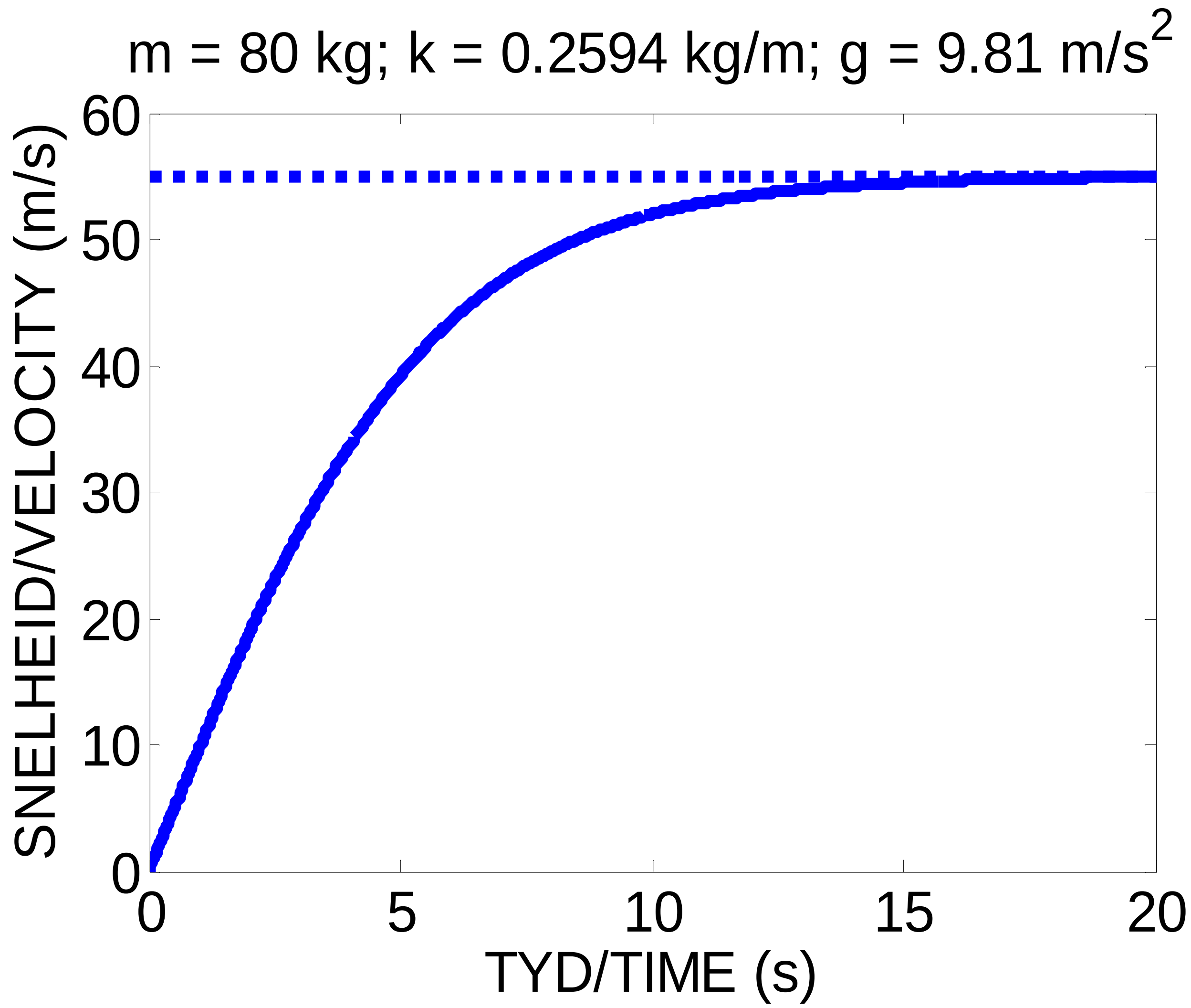

Example: A man in a parachute falls from rest and reaches a limit velocity of 4.9 m/s. How long will it take him to reach 2.45 m/s? Assume that air resistance is directly proportional to the velocity squared and take $g = 9.8$ m/s^2.

<u>Answer:</u> 0.2746 seconds