3.9 Eigenvalues and eigenfunctions *(p 167)*

Problem: Given the DE,

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < L,$$

with boundary conditions $y(0) = 0$ and $y(L) = 0$, do constants λ exist for which the DE has a non-trivial solution?

Note: The trivial solution is $y \equiv 0$

Case I: $\boxed{\lambda = 0}$

$$\frac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = c_1 \quad \Rightarrow \quad y = c_1 x + c_2$$

$$y(0) = 0 \quad \Rightarrow \quad c_2 = 0$$

$$y(L) = 0 \quad \Rightarrow \quad c_1 = 0$$

$$\Rightarrow \quad y \equiv 0 \quad \Rightarrow \quad \text{trivial solution!}$$

$$\Rightarrow \quad \lambda = 0 \quad \text{does not work!}$$

Case II: $\boxed{\lambda < 0}$

$$\frac{d^2y}{dx^2} = \overbrace{-\lambda}^{>0} y$$

$$y(x) = c_1 \cosh(\sqrt{-\lambda}\, x) + c_2 \sinh(\sqrt{-\lambda}\, x)$$

$$y(0) = 0 \quad \Rightarrow \quad c_1 = 0$$

$$y(L) = 0 \quad \Rightarrow \quad c_2 = 0$$

$\Rightarrow \quad y \equiv 0 \quad \Rightarrow$ trivial solution!

$\Rightarrow \quad \lambda < 0$ does not work!

Case III: $\boxed{\lambda > 0}$

$$\frac{d^2y}{dx^2} = \overbrace{-\lambda}^{<0} y$$

$$y(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

$$y(0) = 0 \quad \Rightarrow \quad c_1 = 0$$

$$y(L) = 0 \quad \Rightarrow \quad 0 = c_2 \sin(\sqrt{\lambda}\, L)$$

Now $\sin(\sqrt{\lambda}\, L)$ can be zero however!

$$\Rightarrow \quad \sqrt{\lambda}\, L = n\pi, \quad n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, 3, \ldots$$

$$\Rightarrow \quad y_n = c_2 \sin\left(\frac{n\pi}{L}\, x\right), \quad n = 0, 1, 2, 3, \ldots$$

Note: $n \neq 0$ (for $n = 0$, the trivial solution is obtained!)

In summary: The DE

$$\frac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < L, \quad y(0) = 0, \quad y(L) = 0$$

has non-trivial solutions only when $\lambda = \lambda_n$, where

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$$\lambda_n = \frac{n^2\pi^2}{L^2},\ n = 1, 2, 3, \ldots$$ **(Eigenvalues)**

$$y_n = C\,\sin\left(\frac{n\pi}{L}x\right),\ n = 1, 2, \ldots$$ **(Eigenfunctions)**

The const C can take any value, as long as $C \neq 0$

Take $C = 1$ and sketch y_1, y_2, and y_3