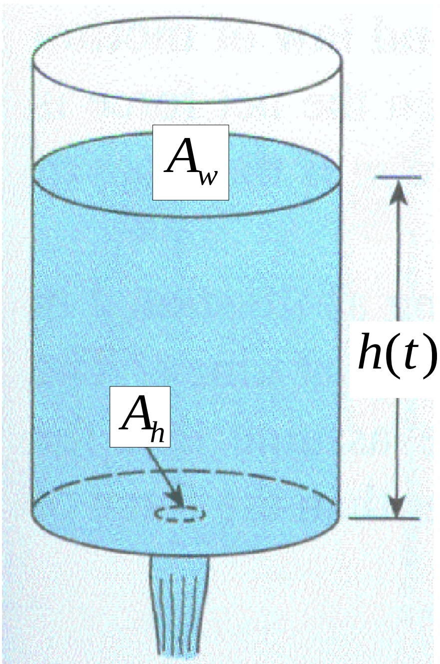
Application: Flow of water from a tank

(see prob 13, p 88 and eqn (10), p 22)



Consider a cylindrical tank, initially filled to a height of h_0. Water runs out through a hole in the bottom of the tank. Find the height of the water level at any time instant t.

Let $h = h(t)$ be the height of the water level at time t, A_w the surface area of the water level, A_h the surface area of the hole, and g gravitational acceleration.

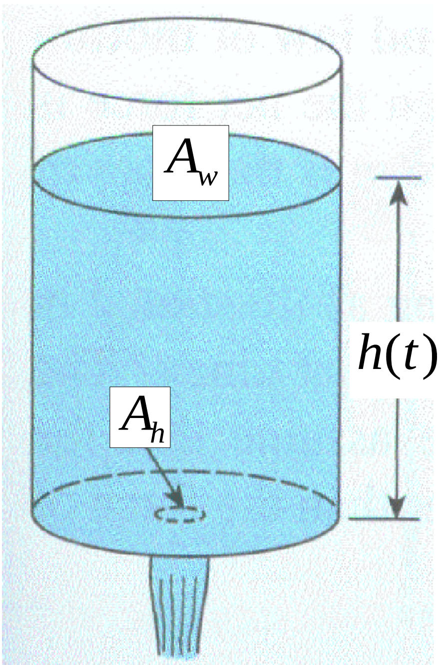
Assumptions:
- Ignore friction and contraction of water at hole
- Toricelli's law (page 21)

Toricelli's law (page 21)

Each water droplet squirts through the hole at the same velocity v it would acquire in falling freely (from rest) from a height h

$$\Rightarrow \quad v(h) = \sqrt{2gh}$$

<u>Formulate:</u> Let $V = V(t)$ be the volume of the water in the tank at time t



Water level drops Δh in time Δt

$$\Delta V = -\left(\sqrt{2gh} \cdot \Delta t\right) A_h$$

$$\frac{\Delta V}{\Delta t} = \frac{dV}{dt} = -A_h\sqrt{2gh}$$

$$V = A_w h$$

$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2g}\, h^{1/2}$$

$$\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2g}\, h^{1/2} = -\rho\sqrt{2g}\, h^{1/2}, \quad h(0) = h_0$$

Drainage of a cylindrical tank

Solution: Integration factors will not work, since DE is non-linear

Therefore use separation of variables and show that:

$$h(t) = \left({h_0}^{1/2} - \rho \sqrt{\frac{g}{2}}\, t \right)^2$$

The tank is empty when $h = 0$, therefore when

$$t = \frac{{h_0}^{1/2}}{\rho} \sqrt{\frac{2}{g}}$$

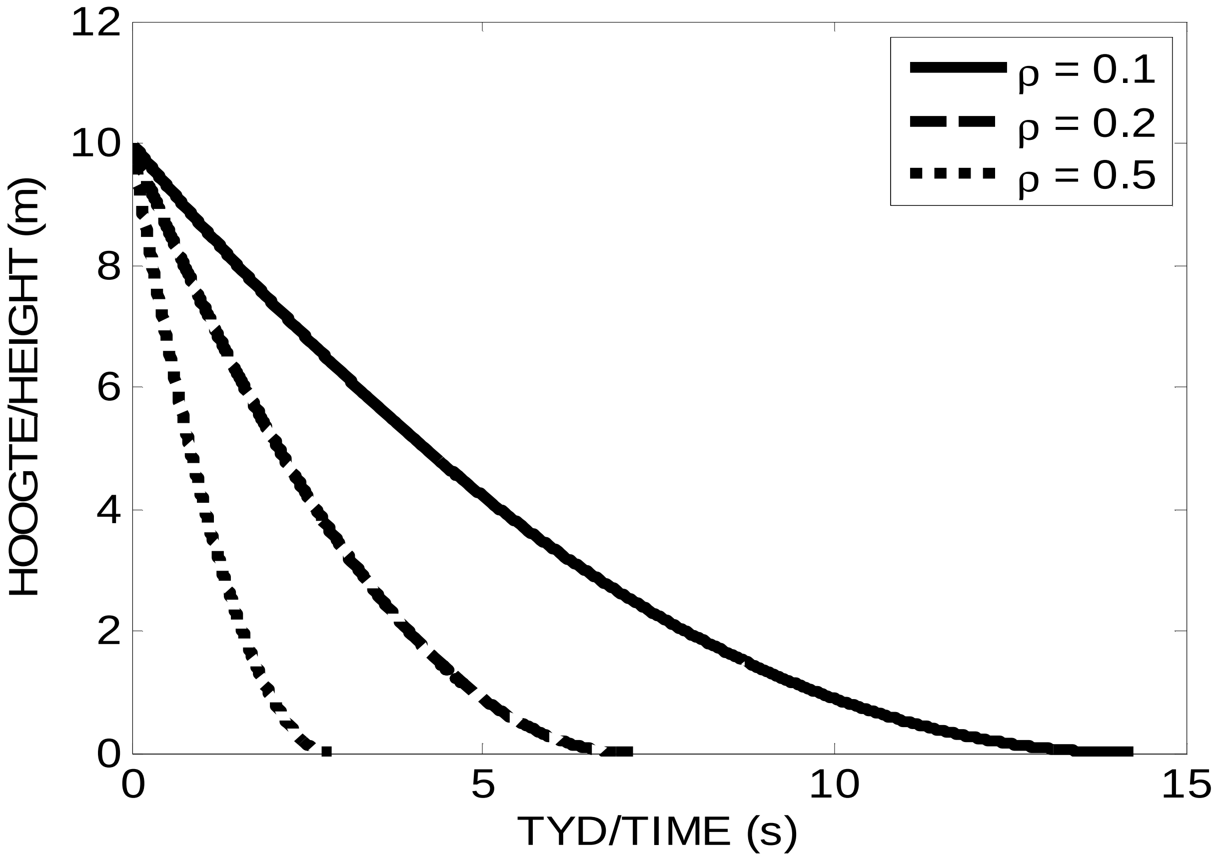
$$\text{Therefore:} \quad t \in \left[0,\ \frac{{h_0}^{1/2}}{\rho} \sqrt{\frac{2}{g}} \right]$$

Show that the unit for $\dfrac{{h_0}^{1/2}}{\rho} \sqrt{\dfrac{2}{g}}$ is seconds

Test: Suppose that the tank is entirely open at the bottom, then $h_0 = \dfrac{1}{2} g t^2$ and $t = \dfrac{{h_0}^{1/2}}{1} \sqrt{\dfrac{2}{g}} \Rightarrow \rho = 1$

Drainage of a cylindrical tank

Example ($h_0 = 10\,\text{m}$; $g = 9.81\,\text{m/s}^2$)



Graph for general case:

$\dfrac{dh}{dt}$ is always negative $\Rightarrow$ graph is always decreasing

Show that $\dfrac{d^2h}{dt^2} = \rho^2 g > 0 \Rightarrow$ concave upwards