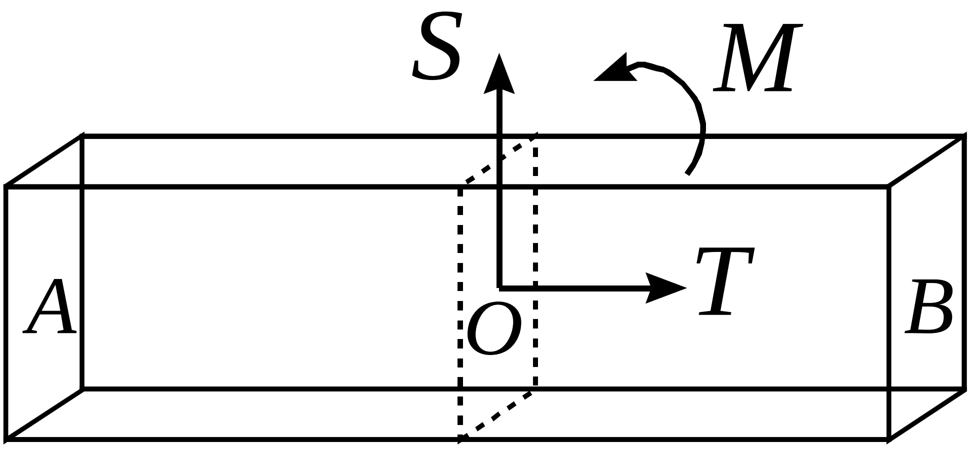
3.9 Deflection of a loaded beam *(page 165)*

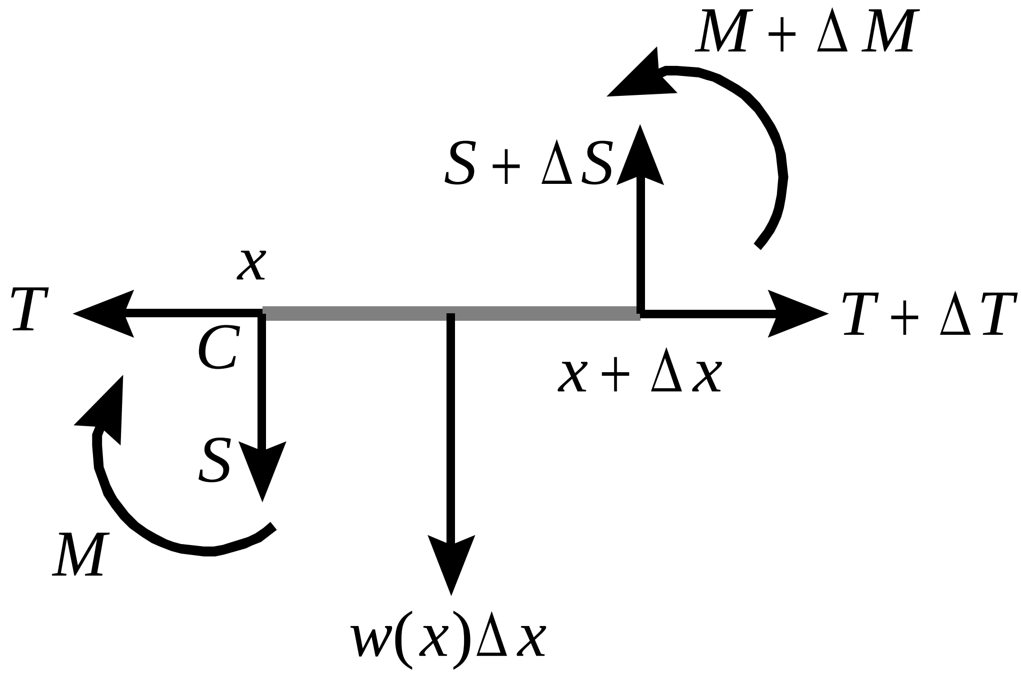
Problem: Determine the deflection of a beam as in fig 3.9.2 in Z&W (see supplementary notes on web)



Consider a section O. Part B exerts a tension force T, shearing force S, and bending moment M onto part A. Choose x-axis $\rightarrow$ and y-axis $\uparrow$.

Assume:
(1) Beam is loaded with weight $w(x)$ per unit length
(2) No external horizontal forces

Now consider small element with length Δx:



For equilibrium we have:

$$\boxed{(\rightarrow)\ \Sigma F_x = 0}\ T + \Delta T - T = 0 \quad (1)$$

$$\boxed{(\uparrow)\ \Sigma F_y = 0}\ S + \Delta S - S - w(x)\Delta x = 0 \quad (2)$$

$$\boxed{\text{(CCW)}\ \Sigma M_C = 0}$$

$$M + \Delta M - M + (S + \Delta S)\Delta x - (w(x)\Delta x)\frac{1}{2}\Delta x = 0 \quad (3)$$

In eqn (3), ignore terms that are small to 2nd order i.e. $\Delta S \Delta x$ and $\Delta x \Delta x$:

$$M + \Delta M - M + S\Delta x = 0 \text{ (3')}$$

In (1), (2) and (3'), cancel terms and divide by Δx. Also let $\Delta x \to 0 \Rightarrow$ 3 basic DEs for beams:

$$\frac{dT}{dx} = 0, \quad \frac{dS}{dx} = w(x), \quad \frac{dM}{dx} = -S$$

To find $y = y(x)$, differentiate the 3rd of the above eqns wrt x, and use the middle eqn to eliminate S:

$$\frac{d^2M}{dx^2} = -\frac{dS}{dx} = -w(x) \text{ (4)}$$

Euler-Bernoulli theorem for thin beams: Assume that $M \propto$ curvature K:

$$M = (EI)K \qquad \Rightarrow \qquad M \approx EI\,\frac{d^2y}{dx^2} \text{ (5)}$$

$E \equiv$ Young's modulus; $I \equiv$ moment of inertia

Deflection of a loaded beam

For derivation of $K \approx \dfrac{d^2y}{dx^2}$, when $\dfrac{dy}{dx}$ is small (i.e. when deflection is small) ...

... see supplementary notes (page 2) (SELF STUDY)

Differentiate (5) twice w.r.t. x, and use (4) to eliminate M. Then follows:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = -w(x)$$

With Z&C's choice of the y-axis downwards and E and I both constant, eqn (4) on page 166 follows:

$$\boxed{EI\frac{d^4y}{dx^4} = w(x)}$$

The above is known as the **BEAM EQN**

Forth order DE: For unique solution **four** boundary conditions are required, two left and two right

If we differentiate repeatedly (with y-axis $\uparrow$ and EI const):

$$
\begin{aligned}
EI\frac{d^2y}{dx^2} &= M \\
EI\frac{d^3y}{dx^3} &= \frac{dM}{dx} = -S \\
EI\frac{d^4y}{dx^4} &= -\frac{dS}{dx} = -w(x)
\end{aligned}
$$

With y-axis $\downarrow$ $\Rightarrow$ $EI\dfrac{d^4y}{dx^4} = \dfrac{dS}{dx} = w(x)$

Typical boundary conditions:

Embedded on both sides

$y(0) = 0,\ y'(0) = 0$ $\qquad$ $y(L) = 0,\ y'(L) = 0$

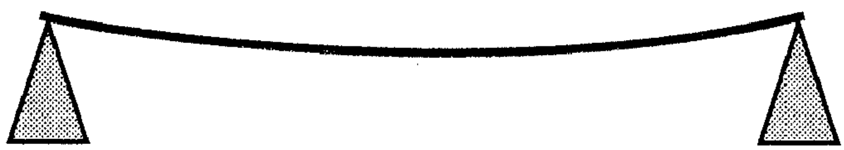
Embedded left and free right

$x = 0$ $x = L$

$y(0) = 0,\ y'(0) = 0$ $y''(L) = 0,\ y'''(L) = 0$

Left: deflection & slope zero Right: M & S zero

Simply supported on both sides



$x = 0$ $x = L$

$y(0) = 0,\ y''(0) = 0$ $y(L) = 0,\ y''(L) = 0$

Left: defl & M zero Right: defl & M zero

Example 1 (page 166): A beam of length L is embedded on both ends. Find the deflection of the beam if a constant load w_0 is uniformly distributed along its length, i.e. $w(x) = w_0,\ 0 < x < L$.

$$EI\frac{d^4y}{dx^4} = w_0$$

Answer: $y(x) = \dfrac{1}{24}\dfrac{w_0}{EI}\,x^2(x-L)^2$

Maximum deflection: $y_{\max} = \dfrac{1}{384}\dfrac{w_0}{EI}\,L^4$

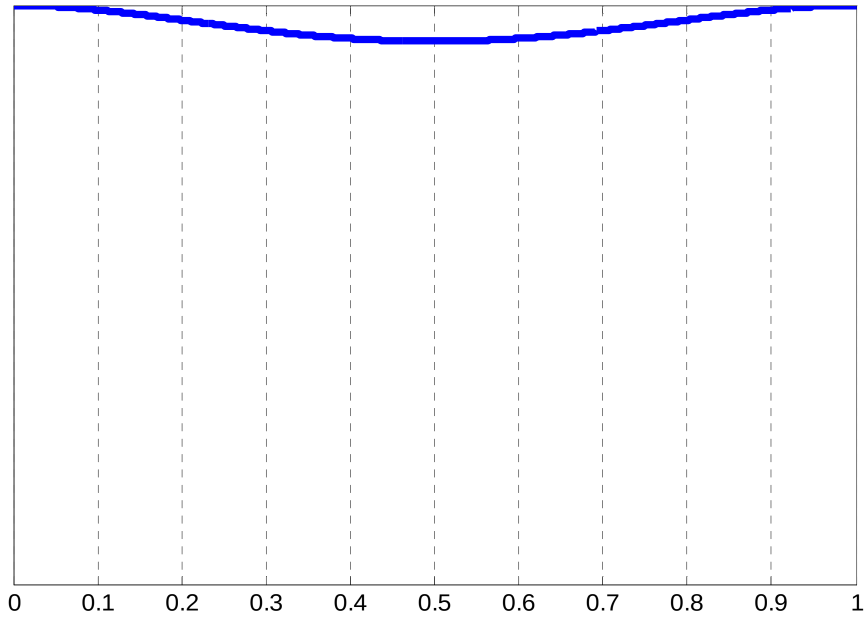
Example 2 (Ex 3.9, Nr 3): Similar to example 1, but embedded left and simply supported right.

Answer: $y(x) = \dfrac{w_0}{48EI}\,x^2(x-L)(2x-3L)$

Example 3: Similar to example 1, but free left and simply supported right.

Answer: Unsolvable

Voorbeeld 1 / Example 1: w_0=1; EI=1; L=1



Voorbeeld 2 / Example 2: w_0=1; EI=1; L=1