2.7: Application 3: Compound interest

Problem: An amount of P_0 is invested at an annual interest rate of x%. How much money is in the account after t years?

Interest rate: $r = x\% = x/100$

<u>With interest compounded annually:</u>

Initially: $P(0) = P_0$
After 1 year: $P(1) = P_0 + rP_0 = (1+r)P_0$
After 2 years: $P(2) = (1+r)P(1) = (1+r)^2P_0$
After 3 years: $P(3) = (1+r)^3P_0$
$\vdots$
After t years: $P(t) = (1+r)^tP_0$

<u>With interest compounded biannually:</u>

Initially: $P(0) = P_0$
After $\frac{1}{2}$ year: $P(\frac{1}{2}) = P_0 + \frac{r}{2}P_0 = \left(1+\frac{r}{2}\right)P_0$
After 1 year: $P(1) = (1+\frac{r}{2})P(\frac{1}{2}) = \left(1+\frac{r}{2}\right)^2P_0$
$\vdots$
After t years: $P(t) = \left(1+\frac{r}{2}\right)^{2t}P_0$

Compound interest

With interest compounded quarterly:

After t years: $P(t) = \left(1 + \frac{r}{4}\right)^{4t} P_0$

With interest compounded daily:

After t years: $P(t) = \left(1 + \frac{r}{365}\right)^{365t} P_0$

With interest compounded continuously:

After t years: $P(t) = \left(1 + \frac{r}{n}\right)^{nt} P_0$, with $n \to \infty$

$$= \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{nt} P_0$$

Set: $y = \left(1 + \frac{r}{n}\right)^{nt}$ and $s = r/n \Rightarrow n = r/s$

$$\Rightarrow \quad y = \left[(1+s)^{1/s}\right]^{rt}$$

If $n \to \infty$, then $s \to 0$ (r is constant)

$$\Rightarrow \quad \lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{nt} = \lim_{s\to 0} \left[(1+s)^{1/s}\right]^{rt}$$

$$= \left[\lim_{s\to 0} (1+s)^{1/s}\right]^{rt}$$

Compound interest Tr 3

Let $f = (1+s)^{1/s}$ so that $\ln f = \dfrac{\ln(1+s)}{s}$

$$\Rightarrow \quad \lim_{s\to 0} \ln f = \lim_{s\to 0} \frac{\ln(1+s)}{s}$$
$$= \lim_{s\to 0} \frac{1}{1+s} \text{ (L'Hopital)}$$
$$= 1$$

$$\Rightarrow \quad e^{\lim_{s\to 0} \ln f} = \lim_{s\to 0} e^{\ln f} = \lim_{s\to 0} f = e^1 = e$$

$$\Rightarrow \quad \lim_{s\to 0} (1+s)^{1/s} = e$$

$$\Rightarrow \quad \lim_{n\to\infty} \left(1+\frac{r}{n}\right)^{nt} = \left[\lim_{s\to 0}(1+s)^{1/s}\right]^{rt}$$
$$= e^{rt}$$

$\Rightarrow$ For interest compounded cont'ly: $\boxed{P(t) = P_0 e^{rt}}$

This satisfies the DE:

$$\frac{dP}{dt} = rP \quad \text{with} \quad P(0) = P_0$$

Compound interest

Example: An amount of R 1000 is invested for 3 years at an annual interest rate of 6%. How much is it worth: (a) when the interest is compounded annually, and (b) when the interest is compounded continuously?

Answers: (a) R 1191 (b) R 1197

(SELF STUDY) **Application 4:** Newton's law of cooling *(pp 20 & 75)*